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Implementing the SST Standard Market Model

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The Standard Market Model of the Swiss Solvency Test (SST) described in [1] defines the capital required by an insurance company to absorb negative financial scenarios. This document derives the basic steps that are needed for the computational implementation of the Standard Market model. It also serves as a technical documentation of our *Python / C++* code of the SST model.

Implementing the Model

The SST uses following notation for the 12 months change in risk bearing capital ("Risikotragendes Kapital", RTK hereafter)

$$\Delta\text{RTK}_{t+1} = \text{RTK}_{t+1} - \text{RTK}_t$$

To avoid ambiguity on the indices t and $t+1$ in the above and in following equations one can think of all changes of a state at time t to occur at $t+1$ (or just right before $t+1$) which results in the state at time $t+1$.
RTK itself is a function of z, a vector of economic variables defined by FINMA (with 82 such factors being defined as of end of 2013).

$$\text{RTK}_t = \text{RTK}(z_t)$$

The factors in z_t include the yield curve, equity prices and FX rates and can be thought of as the economic state at time t.
FINMA's guidelines model the evolution of z_t over time as the random variable X_{t+1}, which stands for period t change.

$$z_{t+1} = z_t + X_{t+1} \tag{1}$$

The capital requirement imposed by FINMA corresponds to the 1% expected shortfall of the change in RTK in 12 months where it is assumed that the change of the factor z over time is a multivariate normal distributed vector X:[1]

$$\text{RTK}_{t+1} = \text{RTK}(z_t + X_{t+1})$$

FINMA's guidelines suggest a second order Taylor expansion of the above expression. Therefore, using first and second derivatives, we get

$$\begin{aligned}\text{RTK}_{t+1} \approx & \text{RTK}(z_t) + \sum_i \frac{\partial \text{RTK}(z_t)}{\partial z_i} X_i \\ & + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 \text{RTK}(z_t)}{\partial z_i \partial z_j} X_i X_j \end{aligned} \tag{2}$$

While the functioning of the Standard Market Model is clearly explained in [1] it is worthwhile to look into the details and the meanings of the variables.

RTK itself is the difference of assets and technical liabilities. Denote by A_i the CHF value of asset or liability i, where a positive sign means an asset and a negative sign a liability. Assets and liabilities add up to the risk bearing capital:

$$\text{RTK}_t = \sum_i A_{t,i}$$

For each asset at time t we can write

$$A_{t,i} = units_i \times p_{t,i}(z_t)$$

Thus, the value of asset i at time t is determined by $p_{t,i}$ (*the price in CHF*) and the units held. Liabilities are modeled as zero-bonds with maturities that

[1] The inclusion of predefined scenarios is discussed further below.

match the duration of the liabilities; as noted before, they have negative signs.

The SST looks at the development of assets' and liabilities' market value after 12 months, which we denote as $t+1$ here. These values can be written as

$$\begin{aligned} A_{t+1,i} &= units_i \times p_{t+1,i}(z_{t+1}) \\ &= units_i \times p_{t+1,i}(z_t + X_{t+1}) \end{aligned} \tag{3}$$

In the above equation $p_{t+1,i}$ is the pricing function of asset i at time $t+1$. It is important to note that the pricing function of asset i may change over time. This is the case for e.g. bonds because time-to-maturity (i.e. duration) decreases over time which has to be reflected in the pricing function. Moreover, (3) also assumes that units held do not change between t and $t+1$. Hence, for dynamic portfolio strategies, modified assumptions have to be made.

Putting these pieces together we obtain

$$\begin{aligned} \Delta \mathsf{RTK}_{t+1} &= \sum_i A_{t+1,i} - A_{t,i} \\ &= \sum_i units_i \times \Delta P_{t+1,i}\,, \end{aligned} \tag{4}$$

with

$$\begin{aligned} \Delta P_{t+1,i} \approx \quad & p_{t+1,i}(z_t) - p_{t,i}(z_t) \\ + \quad & \sum_j \frac{\partial p_{t+1,i}}{\partial z_j} X_j \\ + \quad & \frac{1}{2} \sum_j \sum_k \frac{\partial^2 p_{t+1,i}}{\partial z_j \partial z_k} X_j X_k. \end{aligned} \tag{5}$$

Eqs. (4) and (5) are the key equations in the Swiss Solvency Test and special attention has to be paid to the time indices. In (5) the term $p_{t+1,i}(z_t) - p_{t,i}(z_t)$ corresponds to the price change over time due to the change of the pricing function but without change of the input parameters. In the case of a normal fixed rate coupon bond this is the pull-to-par-effect that comes from shorter time-to-maturity. In the case of non-time-sensitive assets this effect is zero ($p_{t+1,i}(x) = p_{t,i}(x)$ for any x). The sum in (5) represents the first two terms of the Taylor series of $p_{t+1}(z_{t+1})$, where it is important to see that we derive the pricing function at time $t+1$.

For computational modelling purposes it is useful to write eqs. (4) and (5) in a more compact way as

$$\begin{aligned} \Delta \mathsf{RTK}_{t+1} = & \sum_i units_i \times (p_{t+1,i}(z_t) - p_{t,i}(z_t)) \\ & + \sum_i units_i \left(\boldsymbol{\delta}'_{t+1,i} \boldsymbol{X} + \frac{1}{2} \boldsymbol{X}' \boldsymbol{\Gamma}_{t+1,i} \boldsymbol{X} \right), \end{aligned} \tag{6}$$

where $\boldsymbol{\delta}_i$ and $\boldsymbol{\Gamma}_i$ are the first and second order sensitivities of asset i at time $t+1$ towards factors given by the vector $\boldsymbol{X}$, i.e., the gradient and the Hessian of the pricing function with respect to $\boldsymbol{X}$.

Calculating sensitivities

The calculation of factor sensitivities depends on the asset class and its respective pricing function. The specification of different pricing functions goes beyond the scope of this paper. However, we look at two examples that cover a broad set of relevant cases: log assets with time $invariant$ pricing functions and bonds with time $variant$ pricing functions.

Log Asset – time invariant pricing

Log assets are affected by log shocks such as the asset class relevant shock (equity Switzerland, equity Europe, real estate, and hedge funds) and fx shocks if denominated in foreign currency. For a given shock size h the first derivative is given by

$$\begin{aligned} \delta_{t+1,j} &= \frac{\partial p_{t+1}}{\partial z_j} = \frac{\partial p_t}{\partial z_j} \\ &= \wp_{t,i} \frac{\exp(h) - \exp(-h)}{2h} \\ &= \wp_{t,i} \frac{\sinh(h)}{h}\,, \end{aligned} \tag{7}$$

where $\wp_{t,1}$ is the observable market price of asset i at time t in Swiss francs (we use script $\wp$ for the price to distinguish it from the pricing function p). In (7) we use the fact that log assets' pricing functions do not change over time ($p_t(z) = p_{t+1}(z)$) and that for log assets the price changes to $\wp_t \times \exp(h)$ after a log shock of size h is applied.

The second derivatives in the Taylor-series are also easy to calculate for log assets:

$$\begin{aligned} \Gamma_{t+1,jk} &= \frac{\partial^2 p_{t+1}}{\partial z_j \partial z_k} = \frac{\partial^2 p_t}{\partial z_j \partial z_k} \\ &= \wp_t \frac{\exp(2h) + \exp(-2h) - 2}{4h^2} \\ &= \wp_t \frac{\sinh^2(h)}{h^2}\,. \end{aligned} \tag{8}$$

In (8) we assume for notational simplicity (and also because of FINMA specifications) that $h_j = h_i$ for log assets (basic move ('Basisauslenkung') is identical for equity, real estate, fx, hedge funds etc.).

Bonds – time variant pricing

Some asset types, such as bonds and other interest rate sensitive instruments, have time variant pricing functions. The first derivative $\boldsymbol{\delta}$ is then given by:

$$\begin{aligned}\delta_{t+1,j} &= \frac{\partial p_{t+1}}{\partial z_j} \\ &= \frac{p_{t+1}(z_t + h_j \boldsymbol{e_j}) - p_{t+1}(z_t - h_j \boldsymbol{e_j})}{2h_j} \\ &=: \frac{\wp^{+1}_{t+1} - \wp^{-1}_{t+1}}{2h_j}, \end{aligned} \tag{9}$$

where $\boldsymbol{e_j}$ is the unit vector with 1 at position j and 0 everywhere else (formally, $(\boldsymbol{e_j})_i = 1$ for $i = j$ and $(\boldsymbol{e_j})_i = 0$ for $i \neq j$).
In contrast to the log asset example, we cannot use observed market prices for calculating $\boldsymbol{\delta}$. Instead, we need to calculate *model prices* $\wp$ at time $t+1$. The notation $\wp^{+1}$ and $\wp^{-1}$ in (9) is a shortcut for prices after a positive or negative shock of absolute size h, respectively.
The second derivatives of bonds are calculated as follows:

$$\begin{aligned}\Gamma_{t+1,jk} = \frac{\partial^2 p_{t+1}}{\partial z_j \partial z_k} = \\ \frac{1}{4h_j h_k} \times \\ \sum_{a\in\{-1,+1\}} \sum_{b\in\{-1,+1\}} ab p_{t+1}\left(z_t + a h_j \boldsymbol{e_j} + b h_k \boldsymbol{e_k}\right) \\ = \frac{\sum_{a\in\{-1,+1\}} \sum_{b\in\{-1,+1\}} ab \wp^{ab}_{t+1}}{4h_j h_k}. \end{aligned} \tag{10}$$

The calculation of the second derivative Γ and the model prices with positive and negative shocks $\wp^{ab}$ may be computationally quite time consuming. This is especially the case for bonds with long maturities as they have sensitivities towards many (interest rate-) factors.

Equation (10) is the general notation for calculating the second derivative of a bond towards factors j and k. At the same time it is also useful as guidance for calculating Γ_{ij} for two interest rate shocks i and j. However, if one of the shocks is not an interest rate but a log shock (i.e. fx shock), equation (10) can be written more specifically. We assume shock i is an interest rate shock and shock j an fx shock. Then we can write

$$\begin{aligned}\Gamma_{t+1,jk} &= \frac{\wp^{+1}_{t+1}(\exp(h_j) - \exp(-h_j))}{4h_j h_k} \\ &\quad + \frac{\wp^{-1}_{t+1}(\exp(-h_j) - \exp(h_j))}{4h_j h_k} \\ &= \sinh(h_j)\frac{\wp^{+1}_{t+1} - \wp^{-1}_{t+1}}{2h_j h_k}\end{aligned}$$

If both shocks are log shocks which is the case for the diagonal term Γ_{jj} where j is the fx shock on the currency of the foreign denominated bond, we obtain

$$\begin{aligned}\Gamma_{t+1,jj} &= \wp_{t+1}\frac{\exp(2h_j) + \exp(-2h_j) - 2}{4h_j^2} \\ &= \wp_{t+1}\frac{\sinh^2(h_j)}{h_j^2}, \end{aligned}$$

where $\wp_{t+1}$ is the model price of the unshocked bond at $t+1$ which we denoted $p_{t+1}(z_t)$ before.
As a numerical evaluation of the pricing function may consume a significant amount of time, it is important to know how often it has to be invoked when calculating $\boldsymbol{\Gamma}$. The following list gives an overview of the number of pricing calculations for different shock types (IR: interest rates, FX: foreign currency devaluation/appreciation)

- IR + IR: $\wp^{ab}_{t+1}$, $a \in \{-1,+1\}$, $b \in \{-1,+1\} \Rightarrow 4$ times
- IR + FX: $\wp^{+1}_{t+1}$, $\wp^{-1}_{t+1} \Rightarrow 2$ times
- FX + FX: $\wp_{t+1} \Rightarrow 1$ time

Clearly, the IR+IR case is the most relevant in terms of computational time, as there are $n(n-1)/2 + n$ Gamma terms if n is the number of sensitivities towards interest rates. Note that in the extreme case of a bond with 50 years maturity we have 10 sensitivities towards interest rates according to FINMA specifications plus 1 for credit spread sensitivity giving us 11 interest rate factor sensitivities. As a result for a bond with 50 years maturity we need to calculate the price for different input parameters 264 times to get all interest rate relevant $\boldsymbol{\Gamma}$ values.

Including extreme scenarios

FINMA also specifies a set of extreme market scenarios which have to be included into the calculation of the capital requirements. These scenarios are weighted with predefined probabilities.

Denote by $\Delta\text{RTK}_{t+1,n}$ the *additional loss in CHF* (may theoretically be a gain, too) occurring in scenario n. Let Y_n be a random variable that is 1 if shock n occurs and 0 otherwise. The event that no shock occurs is indexed by $n = 0$, so that $\Delta\text{RTK}_{t+1,0} = 0$. Shocks are mutually exclusive, so that $\sum_{n=0}^{N} \text{Prob}(Y_n = 1) = 1$. The factors for the scenarios predefined by FINMA are denoted by X_n. Including these scenarios the main equation (2) of the Standard Market Model results in

$$\begin{aligned}\text{RTK}_{t+1} \approx \\ &\text{RTK}(z_t) + \sum_i \frac{\partial \text{RTK}(z_t)}{\partial z_i} X_i \\ &+ \frac{1}{2} \sum_i \sum_j \frac{\partial^2 \text{RTK}(z_t)}{\partial z_i \partial z_j} X_i X_j \\ &+ \sum_{n=0}^{N} \Delta\text{RTK}_{t+1,n} Y_n \,,\end{aligned}$$

where

$$\Delta\text{RTK}_{t+1,n} = p_{t+1}(z_t + X_n) - p_t(z_t) \,.$$

Breaking this up for all assets and liabilities we rewrite (6) as

$$\begin{aligned}\Delta\text{RTK}_{t+1} = \\ \sum_i units_i \times \{p_{t+1,i}(z_t) - p_t(z_t) \\ + \boldsymbol{\delta}'_{t+1,i} \boldsymbol{X} + \frac{1}{2} \boldsymbol{X}' \boldsymbol{\Gamma}_{t+1,i} \boldsymbol{X} \\ + \sum_n \left(p_{t+1,i}(z_t + X_n) - p_{t,i}(z_t)\right) \times Y_n\}\end{aligned} \tag{11}$$

Note that the term $p_{t+1,i}(z_t + X_n) - p_{t,i}(z_t)$ is somewhat different from the others. It subtracts an observed price $p_{t,i}$ from a model price $p_{t+1,i}$. If the pricing function p has a systematic error, this error is added to this difference. To avoid this we suggest calculating the model price at time t instead of using observed prices.[2] Systematic errors are likely to be reduced this way.

[2] The bond has obviously an observed price and a model price at time t. We aim to specify the pricing function in a way that both prices are similar. For log assets this is not relevant as $p_{t,i}(z_t) = p_{t+1,i}(z_{t+1})$.

Incorporating Cash Flows

Eq. (3) assumes either that no cash flows occur during t and $t+1$ or that the pricing function does account for this: Obviously, the value of A_i should not drop if the price of an asset drops due to cash flow distribution. Instead, a reinvestment policy of the cash flows has to be defined (or assumed) to calculate the future value of asset i. In the model's December implementation we assume by default that proceedings are held in zero-interest yielding cash.

$$\begin{aligned}\Delta\text{RTK}_{t+1} = &\sum_i units_i \times \{p_{t+1,i}(z_t) - p_t(z_t) + \text{CF}_i \\ &+ \boldsymbol{\delta}'_{t+1,i} \boldsymbol{X} + \frac{1}{2} \boldsymbol{X}' \boldsymbol{\Gamma}_{t+1,i} \boldsymbol{X} \\ &+ \sum_n \left(p_{t+1,i}(z_t + X_n) - p_{t,i}(z_t)\right) \times Y_n\} \,,\end{aligned} \tag{12}$$

where CF_i is the sum of all cash flows between t and $t+1$ of security i. In order to avoid double-counting of cash-flows, special attention has to be paid to the last cash flow and that it is aligned to the security's *ex-date*.

Conclusion

This document shows the steps relevant for implementing the basics of the standard SST Market model. It describes how the delta-gamma model can be specified when the insurance company's balance sheet structure is known.

Target capital is typically calculated by a Monte Carlo simulations of (12) and this may take considerable time especially if it has to be performed repeatedly such as for interactive web-based applications, for the screening of a broad asset universe, or any sort of optimizations. To speed up results we developed an analytical solution that replaces the Monte Carlo approach and speeds up computational time by a factor of several thousands depending on the problem size. For details refer to [2].

In the context of the Swiss Solvency Test and Solvency II, the key challenge remains of course the specification of the asset pricing function, denoted above as $p_t(\cdot)$ and $p_{t+1}(\cdot)$, respectively. With our *SolvencyAnalytics Working Paper Series* we aim to provide transparency on the risk- and asset pricing models we use and to publish regularly analyses and empirical tests on our website.

References

[1] FINMA (2012): Wegleitung zum SST-Marktrisiko-Standardmodell

[2] A. Niedermayer (2013): The Standard Market Risk Model of the Swiss Solvency Test: an Analytical Solution, SolvencyAnalytics Working Paper Series 2013-11 (01)