The Standard Market Risk Model of the Swiss Solvency Test: an Analytical Solution

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Approximately half of the Swiss insurance and reinsurance companies use the standard market model of the Swiss Solvency Test to compute their solvency capital requirement (SCR). While this can be easily implemented using a Monte Carlo simulation the results are either not sufficiently precise or the computational time is long. However, some applications require a repeated computation of the SCR or a fast computation e.g. for an interactive comparison of different investment allocations. Therefore, a fast and precise algorithm is essential for certain applications.
This paper derives such an algorithm for the Swiss Solvency Test's standard market model using a Fast Fourier Transformation of the analytically given characteristic funtion of the target distribution. We compare the results with a Monte Carlo simulation in terms of performance and accuracy. We believe that with this algorithm a broader range of applications is feasible than with traditional Monte Carlo methods.

1 Introduction

The Swiss Solvency Test has been in use since 2011 and defines capital adequacy requirements for Swiss insurance companies. About half of the Swiss insurance and reinsurance companies use the standard model provided by the Swiss Federal Office of Private Insurance and the Swiss Financial Market Supervisory Authority (Finma).
The guidelines of Finma describe a Monte Carlo algorithm for the computation of the Risk Bearing Capital implied by the portfolio of an insurance company. While the Monte Carlo algorithm in the guidelines is easy to implement, it either provides an insufficient level of precision or takes too long to compute for many applications, such as multiple evaluation of the target capital and interactive comparisons. For such applications, a fast and precise evaluation of the target capital is essential. In this article we describe and algorithm based on Fast Fourier Transformation that provides a solution to this problem.

2 Specification of the Swiss Solvency Test

Here we briefly summarize the requirements specified by the Swiss Financial Market Supervisory Authority (Finma).

There is a vector of factors x that influences the development of the prices of assets in an insurance company's portfolio. x is assumed to be normally distributed with mean μ and covariance matrix Σ. As a first step ignore the possibility of "rare events" such as financial crises occurring. In this "normal scenario" the value of an asset j is

$$y_j(x) = \frac{1}{2} x' \Gamma_j x + \delta_j' x \,,$$

where Γ_j and δ_j are the quadratic and linear coefficients of the impact of the risk factors x on the value of the asset y_j. Γ_j and δ_j can be thought of as the first two terms of a Taylor series. See the Finma guidelines on how to compute Γ_j and δ_j for different assets.

A normal distribution does not take into account that "rare events" can occur which lead to exceptionally high losses in financial markets. To take this into account, Finma has amended the model by multiple scenarios, which occur with small probabilities and are mutually exclusive. The probability of scenario i is p_i, where $i = 0$ stands for the "normal scenario" and $\sum_{i=0}^{K} p_i = 1$. For scenarios $i \geq 1$ the value of an asset has to be modified by the additive term $\bar{x}_i' \Gamma_j \bar{x}_i + \delta_j \bar{x}_i =: S_{ij}$, the factor changes for a scenario $\bar{x}_i$ being defined in the guidelines of Finma.

The scenario adjusted value of an asset $Y(x)$ is hence given by

$$Y_j(x) = y_j(x) + \sum_i I_i S_{ij} \,,$$

where the indicator random variables I_i select which scenario occurs. Formally, with probability p_i, $I_i = 1$ and $I_k = 0$ for all $k \neq i$.

What matters at the end of the day is the sum of the changes of values for all assets, i.e.

$$Y(x) = \sum_j Y_j(x) \,.$$

The target capital (TC) is the Expected Shortfall of an insurance company's change in risk bearing capital, i.e. the average loss that occurs with a probability less than 1%. Formally, this is defined as follows. Define the distribution of the variable $Y(x)$ as $F_T(Y)$, where F_T can be computed from the distributions of the random variables x and I_i. Further, define the threshold for the Expected Shortfall y_0 implicitly by $F_T(y_0) = 0.01$. Then the target capital is defined as

$$\text{TC} = \int_{-\infty}^{y_0} y \, dF_T(y) \,.$$

For the following analysis, it is useful to define the sums of Γ_j, δ_j, and S_{ij},

$$\Gamma := \sum_j \Gamma_j, \qquad \delta := \sum_j \delta_j, \qquad s_i := \sum_j S_{ij} \,.$$

Note that for the random variable $Y(x)$ only the sums matter, since

$$y(x) = \frac{1}{2} x' \Gamma x + \delta' x \,,$$

and

$$Y(x) = y(x) + \sum_i I_i s_i \,.$$

Observe that if both Σ and Γ were the identity matrix (or the identity matrix times a constant), y would follow a χ^2 distribution for which efficient computational methods are well known. However, typically, this is not the case and one has to derive an algorithm that goes beyond χ^2 distributions.

In the following we will work with these sums. First, we will consider the quadratic normal random variable $y(x)$ and derive an efficient computational method for it. Then we will describe the computation of the change of values $Y(x)$.

3 Quadratic Normal Random Variable

$x \sim \mathcal{N}(\mu, \Sigma)$, that is the density of x is $p(x) \propto \exp(-\Phi(x))$ where

$$\Phi(x) = \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\,. \tag{1}$$

The quadratic normal random variable y is constructed from x by

$$y(x) = \frac{1}{2}x'\Gamma x + \delta' x\,. \tag{2}$$

One can diagonalize both quadratic forms simultaneously as follows. As a first step, we make a Cholesky decomposition

$$\Sigma = \Lambda\Lambda'\,, \tag{3}$$

and introduce the random variable ξ through

$$x = \Lambda\xi + \mu\,. \tag{4}$$

With this we obtain

$$\Phi(\xi) = \frac{1}{2}\xi'\xi\,, \tag{5}$$

and

$$y(\xi) = \frac{1}{2}\xi'\hat{\Gamma}\xi + \hat{\delta}'\xi + \hat{c}\,, \tag{6}$$

where

$$\hat{\Gamma} = \Lambda'\Gamma\Lambda\,, \tag{7}$$

$$\hat{\delta} = \Lambda'\left(\Gamma\mu + \delta\right)\,, \tag{8}$$

$$\hat{c} = \frac{1}{2}\mu'\Gamma\mu + \delta'\mu\,. \tag{9}$$

Now one can find an orthogonal transformation which diagonalizes $\hat{\Gamma}$:

$$O'\hat{\Gamma}O = D\,. \tag{10}$$

Here O is an orthogonal matrix and D is a diagonal one, with diagonal element d_k. Assume for the moment that $d_k \neq 0$ (but can have any signs).
Introducing

$$\xi = O\eta\,, \tag{11}$$

we have

$$\Phi(\eta) = \frac{1}{2}\eta'\eta = \frac{1}{2}\sum_k \eta_k^2\,. \tag{12}$$

and

$$y(\eta) = \frac{1}{2}\sum_k d_k(\eta_k - \tilde{\mu}_k)^2 + \tilde{c}\,. \tag{13}$$

Introducing

$$\tilde{\delta} = O'\hat{\delta}\,, \tag{14}$$

one has

$$\tilde{\mu}_k = -\frac{\tilde{\delta}_k}{d_k}\,, \tag{15}$$

$$\tilde{c} = \hat{c} - \frac{1}{2}\sum_k \frac{\tilde{\delta}_k^2}{d_k}\,. \tag{16}$$

Here we assumed that all diagonal elements of D are non-zero. If some of the d_k values are zero, then this amounts to adding a normally distributed variable to the weighted sum of (non-central) chi-squared variables. In this case the index k in eqs. (13), (15) and (16) is taken from the set with $d_k \neq 0$, formally $\mathbb{K} := \{k|d_k \neq 0\}$. Define the complementary set as $\overline{\mathbb{K}} := \{k|d_k = 0\}$. One has to add an extra random variable, distributed by standard normal distribution, multiplied by the coefficient

$$b = \left(\sum_{k \in \overline{\mathbb{K}}} \tilde{\delta}_k^2 \right)^{1/2} . \tag{17}$$

Naturally, $\tilde{\mu}_k$ is only defined for $k \in \mathbb{K}$. Further, $\tilde{c}$ is defined as

$$\tilde{c} = \hat{c} - \frac{1}{2} \sum_{k \in \mathbb{K}} \frac{\tilde{\delta}_k^2}{d_k} .$$

The distribution of y can be obtained by a Fourier transformation, and the corresponding code is available in Fortran.

3.1 Distribution of weighted chi-squared variables

The distribution of y is given by

$$f(y) = \int \prod_{k \in \mathbb{K}}^{n} \left(\frac{\mathrm{d}x_k}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x_k^2} \right) \delta \left(\frac{1}{2} \sum_{k \in \mathbb{K}} d_k (x_k - \tilde{\mu}_k)^2 + b x_0 - y \right) , \tag{18}$$

(In a previous version, we changed the notation to $d_k \to a_k$, $\tilde{\mu}_k \to \mu_k$, and b is given by eq. (17). We have made the notation consistent, but some instances of the old notation may still be there.)

Introducing the characteristic function (the Fourier transform) of $f(y)$

$$\hat{f}(t) = \int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{ity} f(y) , \tag{19}$$

one has

$$f(y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} \mathrm{e}^{-ity} \hat{f}(t) . \tag{20}$$

Note that

$$\delta(y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} \mathrm{e}^{-ity} . \tag{21}$$

With this one obtains

$$\hat{f}(t) = \psi(t; b) \prod_{k \in \mathbb{K}} \phi(t; \tilde{\mu}_k, d_k) , \tag{22}$$

where

$$\begin{aligned} \phi(t; \mu, a) &= \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2 + \frac{i}{2} t a (x - \mu)^2 \right) \\ &= \frac{1}{\sqrt{1 - ita}} \exp\left(\frac{1}{2} \mu^2 \frac{ita}{1 - ita} \right) , \end{aligned} \tag{23}$$

and

$$\psi(t; b) = \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2 + itbx \right) = \exp\left(-\frac{1}{2} t^2 b^2 \right) . \tag{24}$$

Note that $\phi(t;\mu,a)$ goes to zero only as $|t|^{-1/2}$, while $\psi(t;b)$ goes to zero exponentially fast. For $b \neq 0$ the integral in (20) converges very fast, while for $n \geq 3$ it converges absolutely even for $b = 0$. In (20), (22) there is no restriction on the signs of the coefficients d_k, the integration can be performed along the real axis of t. (Note, however, that special care should be taken when integrating a strongly oscillating function.)
Although it is not needed for the actual calculation, one can give a rough estimate of the effective range of y. With the notations of (18) we have

$$E((x-\mu)^2) = 1 + \mu^2\,, \qquad E((x-\mu)^4) = 3 + 6\mu^2 + \mu^4\,, \tag{25}$$

$$E(x) = 0\,, \qquad E(x^2) = 1\,. \tag{26}$$

The expectation value of y is given by

$$E(y) = \frac{1}{2}\sum_k d_k(1 + \tilde{\mu}_k^2)\,. \tag{27}$$

The fluctuations are not symmetric, therefor it is useful to consider separately the $a > 0$ and $a < 0$ terms and define

$$\delta y_+ = \left(\frac{1}{2}\sum_{a>0} d_k^2(1 + 2\tilde{\mu}_k^2) + b^2\right)^{1/2}\,, \tag{28}$$

$$\delta y_- = \left(\frac{1}{2}\sum_{a<0} d_k^2(1 + 2\tilde{\mu}_k^2) + b^2\right)^{1/2}\,. \tag{29}$$

One can restrict the range of y to a finite region $y \in [y_a, y_b]$:

$$y_a = \min\{0, E(y) - \alpha\delta y_-\}\,, \quad y_b = \max\{0, E(y) + \alpha\delta y_+\}\,, \tag{30}$$

where, say, $\alpha = 5$. Outside this region the probability is extremely small. We shall, however, determine the range of y not from these relations, but directly from $f(y)$.

3.1.1 Some special cases

For $d_k = a$, $\tilde{\mu}_k = 0$, $b = 0$ and even n it is easy to calculate $f(y)$ analytically. One has (actually for arbitrary positive integer n)

$$f_n(y;a) = \frac{\Theta(y/a)}{|a|\Gamma(n/2)}\left(\frac{y}{a}\right)^{n/2-1} \mathrm{e}^{-y/a}\,. \tag{31}$$

For the case $d_k = a$, $\tilde{\mu}_k \neq 0$, $b = 0$ one has

$$f(y) = \mathrm{e}^{-\mu^2/2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mu^2}{2}\right)^k f_{n+2k}(y;a) \tag{32}$$

where $\mu^2 = \sum_k \tilde{\mu}_k^2$.
For $n_1 = 2\nu_1$ positive eigenvalues a_1 and $n_2 = 2\nu_2$ negative eigenvalues $-a_2$ (and $\tilde{\mu}_k = 0$, $b = 0$) one has

$$\begin{aligned} f(y) = {}& \Theta(-y)\left(\frac{a_2}{a_1+a_2}\right)^{\nu_1}\sum_{k=0}^{\nu_2-1}\frac{\Gamma(\nu_1+k)}{\Gamma(\nu_1)\,k!}\left(\frac{a_1}{a_1+a_2}\right)^k f_{n_2-2k}(-y;a_2) \\ & + \Theta(y)\left(\frac{a_1}{a_1+a_2}\right)^{\nu_2}\sum_{k=0}^{\nu_1-1}\frac{\Gamma(\nu_2+k)}{\Gamma(\nu_2)\,k!}\left(\frac{a_2}{a_1+a_2}\right)^k f_{n_1-2k}(y;a_1) \end{aligned} \tag{33}$$

These analytic results can be used for testing the numerical evaluation of the Fourier transformation.

3.2 Evaluating the Fourier integral

In the integral (20) we restrict the integration to a finite range $t \in [-\tau, \tau]$, choosing τ sufficiently large. We choose τ such that $|\hat{f}(\tau)| \lesssim 10^{-6}$. (The small correction due to the finite interval is taken into account below.)
To evaluate the Fourier integral by FFT we discretize first the integral taking

$$\delta t = \frac{\tau}{M}\,, \tag{34}$$

$$\int_{-\tau}^{\tau} \mathrm{d}t \hat{f}(t) \mathrm{e}^{-iyt} \approx \delta t \sum_{k=-M}^{M} \hat{f}(t_k) \mathrm{e}^{-iyt_k}$$
$$= \delta t \left(-\hat{f}(0) + 2\mathrm{Re} \sum_{k=0}^{M} \hat{f}(t_k) \mathrm{e}^{-iyt_k} \right)\,, \tag{35}$$

where

$$t_k = k\delta t = k\frac{\tau}{M}\,. \tag{36}$$

In order to use the FFT we take N to be a power of 2 and $N > M$ (oversampling) and consider a discrete set of of y values[1]

$$y_j = j\delta y\,, \qquad j = 0, \ldots, N-1 \tag{37}$$

$$\delta y = \frac{2\pi M}{N\tau}\,. \tag{38}$$

Taking $N \gg M$ one can make δy sufficiently small.
Padding $\hat{f}(t_k)$ with zeros we introduce

$$h_k = \begin{cases} \hat{f}(t_k)\,, & \text{for } 0 \le k \le M\,, \\ 0 & \text{for } M+1 \le k \le N-1\,, \end{cases} \tag{39}$$

and obtain for the sum in (35)

$$\sum_{k=0}^{M} \hat{f}(t_k) \mathrm{e}^{-iy_j t_k} = \sum_{k=0}^{N-1} h_k \exp\left(-i\frac{2\pi}{N} jk\right)\,. \tag{40}$$

The sum on the rhs. is the discrete Fourier transformation which can be evaluated by FFT. This gives

$$f(y_j) \approx \frac{\delta t}{2\pi} \left(-\hat{f}(0) + 2\Re\left\{\mathrm{DFT}[h_0, h_1, \ldots, h_{N-1}]\right\}\right)\,. \tag{41}$$

To obtain $f(y)$ for negative y one can rewrite (20) to obtain

$$f(-y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} \mathrm{e}^{-ity} \hat{f}(-t)\,, \tag{42}$$

and using it for $y > 0$ repeat the steps discussed above.

[1] We consider the case of positive and negative values of y separately.

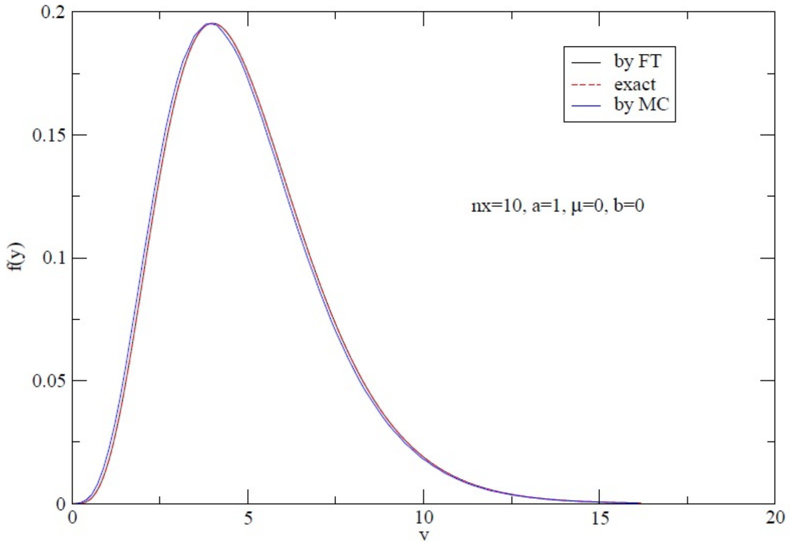

Figure 1: $n = 10,\ d_k = 1,\ \tilde{\mu}_k = 0,\ b = 0.$

3.3 Corrections

The discretization of the integral used in (35) is a good approximation only for $j \ll N/(2\pi)$, i.e. $y \ll M/\tau$. Taking M/τ large enough this will cover the required range of y. It is better, however, to use a corrected formula, obtained by integrating a smooth interpolation of $\hat{f}(t_k)$ (cf. Numerical Recipes).
Denoting $\theta = y\delta t$ one obtains

$$f(y_j) \approx \frac{\delta t}{2\pi}\Big[w(\theta)\left(-\hat{f}(0) + 2\Re\left\{\mathrm{DFT}[h_0, h_1, \ldots, h_{N-1}]\right\}\right) \\ + 2\Re\left\{\mathrm{e}^{-i\omega\tau}\left(\alpha_0(\theta)h_M + \alpha_1(\theta)h_{M-1} + \ldots\right)\right\}\Big]. \quad (43)$$

Taking τ large enough to have $|\hat{f}(\tau)| \ll 1$ one can neglect the error coming from restricting the Fourier integral to a finite range $t \in (-\tau, \tau)$. However, a small correction can be taken into account (approximating the tail of $\hat{f}(t)$ by an exponential function)

$$\int_{-\infty}^{\infty} \mathrm{e}^{-iyt}\hat{f}(t)dt \approx \int_{-\tau}^{\tau} \mathrm{e}^{-iyt}\hat{f}(t)dt + \frac{1}{\pi}\Re\left\{\mathrm{e}^{-iy\tau}\frac{\hat{f}^2(\tau)}{iy\hat{f}(\tau) - \hat{f}'(\tau)}\right\}. \quad (44)$$

4 Checking the results

In special cases the distribution $f(y)$ can be calculated analytically (cf. e.g. (31)).
For the general case, it can be easily simulated by Monte Carlo, although that takes considerably more time if small error is needed.
Some checks are shown in Fig. 1 and Fig. 2.

5 Cumulative Distribution Function

We also need the cumulative distribution function $F(y)$. This will be determined in the discrete points $y_j = j\delta y$ by integrating a cubic polynomial approximation to $f(y_j)$,

$$F(y_{j+1}) = F(y_j) + \frac{13}{24}\left(f(y_j) + f(y_{j+1})\right) - \frac{1}{24}\left(f(y_{j-1}) + f(y_{j+2})\right). \quad (45)$$

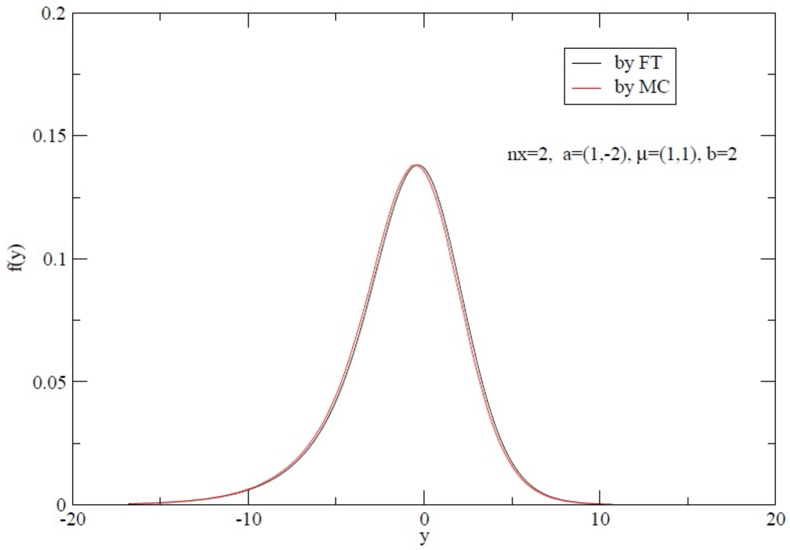

Figure 2: $n = 2,\ a = (1, -2),\ \mu = (1, 1),\ b = 2$

Note that alternatively one can also calculate $F(y)$ by the Gil-Pelaes formula
(cf. http://www.statsresearch.co.nz/robert/QF.htm)

$$F(y) = \frac{1}{2} - \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi t} \mathrm{Im}\left(\mathrm{e}^{-ity} \hat{f}(t)\right) . \tag{46}$$

Our analysis indicates, however, that the discretization errors are larger for this choice.

6 Multiple scenarios

We discuss here a model with the random variable

$$Y = y + u\,, \tag{47}$$

where y is distributed according to (18) and u is independent of y and has a probability distribution

$$f_c(u) = p_0\delta(u) + \sum_{i=1}^{K} p_i\delta(u - s_i)\,, \tag{48}$$

where

$$p_0 + \sum_{i=1}^{K} p_i = 1\,, \tag{49}$$

and δ is the Dirac delta function.
Here i labels the different scenarios, $i = 0$ corresponding to a “normal year” (the shift u is $s_0 = 0$).
The total probability distribution of Y is given by

$$f_T(Y) = \int dy\, du\, f(y) f_c(u) \delta(y + u - Y) = \sum_{i=0}^{K} p_i f(Y - s_i)\,. \tag{50}$$

Since there is a small number of scenarios it is easy to calculate $f_T(Y)$ once $f(y)$ is known.

7 Putting the pieces together

Once one can calculate $f_T(Y)$ efficiently, it is easy to compute the target capital TC. Define the cutoff y_0 as the first percentile, i.e. $F_T(y_0) = 0.01$. Then the target capital can be computed as

$$\text{TC} = \int_{-\infty}^{y_0} y dF_T(y)\,. \tag{51}$$

The integral can be computed efficiently using standard numerical analysis techniques such as Newton-Cotes formulas or Gaussian quadrature.

8 Conclusions

We have shown a fast and precise algorithm for the computation of the target capital. While more thorough performance tests have to be conducted, initial test suggest that our algorithm outperforms the standard Monte Carlo algorithm by a factor 200. This increase in speed opens the possibility for new applications of the Swiss Solvency Test, such as computation of the required target capital for different potential market developments and the interactive comparison of the impact of different investment strategies on the required target capital.