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A Partial Internal Model for Credit and Market Risk Under Solvency II

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Contents

1 Introduction **4**

2 The Solvency II framework **5**

2.1 Characteristics and risks of European life insurance companies 5

2.2 An overview of Solvency II 9

2.3 Credit and market risk in the standard approach 12

3 Credit risk models **17**

3.1 Structural models 18

3.2 Reduced-form models 26

4 The model and its implementation **34**

4.1 Lando (1998)'s model and some extensions 36

4.2 Implementing the model 41

5 Conclusion **48**

References **50**

Appendix - Derivation of the zero-coupon bond price **52**

List of Figures

1 Balance sheet of a representative European life insurer 7
2 Rating and time to maturity of the fixed income portfolio 8
3 Overall structure of the SCR 11
4 An example of a one-year rating transition matrix 20
5 Average cumulative default rates 21
6 The CreditMetrics framework 22
7 An example histogram of the 1-year forward prices and changes in the value of a BBB-rated bond 23
8 Valuation of a single cash flow subject to default risk 25
9 Prices of a bond with different credit ratings 47
10 Prices of bonds with ascending maturities 48
11 Distribution of the portfolio value in one year 49

1 Introduction

With the introduction of Solvency II currently scheduled to come into effect on 1 January 2016, European insurers will probably face the greatest regulatory reform they have ever witnessed. The revision of the current prudential regulation aims at establishing a harmonised, risk-sensitive solvency supervision across the European Economic Area, based upon the actual risk profile of each individual insurance company to promote transparency, international competitiveness, and the protection of policyholders and beneficiaries by enhancing the financial stability of insurance companies.

Besides qualitative expectations and disclosure criteria, the most important requirement from insurance undertakings is the compliance with the Solvency Capital Requirement, which is calibrated to ensure that an insurer's capital will only be insufficient to absorb unexpected losses within a one-year horizon in 1 out of 200 cases. Although a basic calculation methodology of the Solvency Capital Requirement, the so-called standard formula is fixed in legislation, insurers have the option to choose between different methods: besides the standard formula provided by the regulator, they are allowed to use the standard model with a partial internal model taking only a few risk parts into consideration, or they may also choose to develop a full internal model with addressing all the risks the insurance company is exposed to. Actually, they are particularly motivated by the conservative specification of the standard approach to build internal models which appropriately grasp risks inherent in their operation since, in general, this reduces their capital requirement.

This thesis contributes to the preparation for the new regulation by setting itself a target of providing a partial internal model for the risks of the fixed income portfolio of a life insurance undertaking. With this end in mind, we are looking for a model which can properly capture credit risk faced by insurers, and which is also flexible enough to allow for the inclusion of various other risk sources, especially market risk factors.

The structure of the thesis is the following. After introducing the main characteristics and risks of life insurance companies, section 2 answers the question why a new regulatory framework was inevitable, and exhibits its features. In addition, the treatment of credit and market risk by the standard approach is also described in this section in a detailed manner, since in order to develop an internal model focusing on these specific risk categories, it is necessary to be certain about the methodology of the standard approach, as well as to be fully aware of the requirements of the regulation. In section 3 we review the most important types of credit risk models discussed in the literature, compare them and assess their features according to the requirements of this specific problem. In section 4, we firstly detail the considerations behind the choice of the appropriate model, then present the chosen framework. The model is a reduced-form credit risk model introduced by Lando (1998), where credit rating migrations are also taken into account, moreover, by allowing the elements of the matrix representing the process behind credit rating transitions to be stochastic, it provides an opportunity to incorporate market risk factors into the model, as well as to assume a rich variety of dependence structures among exposures. We also emphasize the required extensions of the original model in order to be able to use it for risk management purposes. In the second part of section 4 we implement a simplified specification of the suggested general framework, and exhibit its results. Section 5 concludes the paper.

I would like to express my gratitude first of all to my supervisor, Dániel Havran, who greatly facilitated my work with his ideas and advices. I also would like to thank Daniel Niedermayer from Solvency Analytics, who suggested the topic and provided to me the most important data. Last but not least, I am thankful to György Michaletzky, too, who answered my methodological questions in a cordial and enthusiastic manner. All the remaining mistakes and shortcomings are my own.

2 The Solvency II framework

Insurance companies in the European Economic Area will probably face the greatest regulatory reform they have ever witnessed with the introduction of Solvency II, an economic risk-based regulatory framework. The most important goal of the new framework is to protect policyholders and beneficiaries by enhancing the financial stability of insurance companies. In addition, the new regulation enables to harmonise insurance legislations within Europe with the incentive of enhancing fair competition and increasing transparency (Höring, 2012). Apart from the harmonisation of regulatory approaches towards insurers throughout Europe, however, what were the main reasons for coming up with a completely new way of assessing the solvency of insurance companies? What are the most important characteristics of European life insurance undertakings, and what kinds of risks do they bear which have not been allowed for in the previous regulation? Since the main objective of this study is to develop a partial internal model for the better assessment of credit and market risks for life insurance companies under the requirements of Solvency II, these questions are of particular importance. Thus, this section first describes the main characteristics of a representative European-based life insurer, including its balance sheet structure and the composition of its asset portfolio. Secondly, we present the risks inherent in the operation of insurers, and the change that has recently taken place in their business. We then explain why the introduction of a new regulatory framework has become inevitable, and exhibit its main features. Furthermore, a detailed description is provided about how Solvency II is dealing with credit and market risk. This section finally discusses what features of the new regulation are particularly important for modelling credit and market risk in a way that is consistent with Solvency II.

2.1 Characteristics and risks of European life insurance companies

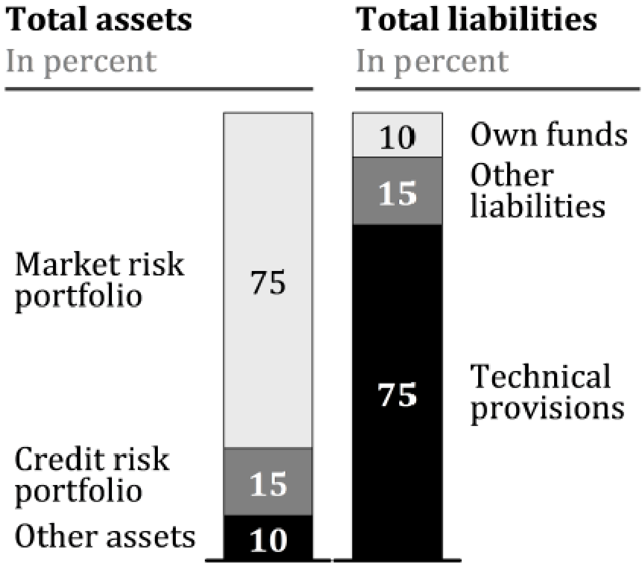
To gain understanding of the motivation for the introduction of the new regulatory environment, it is necessary to have a look at the business model and the resulting balance sheet of a typical European life insurance company. Thus, in the next paragraphs, we show the features of a representative European-based life insurer's balance sheet and investment portfolio as determined by Höring (2012), who created these based on several sources, e.g. statistics from regulators, insurance associations and rating agencies, as well as individual insurance companies' annual reports and investor presentations, taking the requirements of the Solvency II standard model and the Standard & Poor's rating model into account.

According to Höring (2012), the representative life insurance company provides traditional life insurance products and does not engage in either unit-linked products, asset management or banking activities. The role of traditional insurance contracts is to encourage policyholders to reduce their risks by transferring them to enterprises more willing to bear them, in exchange for a premium. The premium is calculated based on the equivalence principle: the expected present value of claims should be equal to the expected present value of premiums. The proceeds from the premium payments (after the deduction of various costs) are then invested in order to create technical provisions for the future claim payments. This feature makes insurance companies financial intermediaries: the premiums collected from policyholders are invested on financial markets (Lorent, 2008). Thus, insurance companies are the largest institutional investors in European financial markets. Figure 1 shows the balance sheet structure of the representative European life insurer (based upon the average market value balance sheet of life insurance companies according to QIS5), the logic behind which is now straightforward. Total assets are assumed to be equal to €4 billion. The market risk portfolio of the asset side consists of investments relevant for the market risk modules in Solvency II, such as equities, real estate properties and debt instruments. In the same vein, the credit risk portfolio includes mortgages, reinsurance assets, loans and cash held at banks, which are relevant for the counterparty default risk module of the regulation. On the liability side, besides technical provisions which are the main source of funding for insurance companies, other liabilities include for example short-term debt and deferred taxes. Own funds usually make up about 10% of total liabilities, which makes the insurance undertaking a highly leveraged enterprise. The insurance liabilities have a duration of 8.9 years (which is the median duration for life insurance liabilities in Europe according to the results of QIS4 (Höring, 2012)). The duration mismatch between assets and liabilities is 2.4 years, which means that a parallel downward shift in the yield curve decreases the company's own funds since technical provisions have a higher duration (Höring, 2012).

The market risk portfolio contains equities and alternatives (private equity, infrastructure and renewable energy projects and hedge funds), real estate properties and debt instruments. Since the technical interest rate[1] is a guaranteed minimal return needed to be provided to the policyholder, in addition, the return above the technical interest rate from the investment of the technical provisions is usually in large part returned to the policyholder (via an enhanced amount of

[1]The technical interest rate is a discount rate used to calculate the premium of the life insurance contract by the equivalence principle, and also a guaranteed minimal return on the invested technical provisions provided to the policyholder. Since the higher the technical interest rate, the lower the premium of the contract, the maximum of the technical interest rate is fixed by legislation.

Figure 1: *Balance sheet of a representative European life insurer*



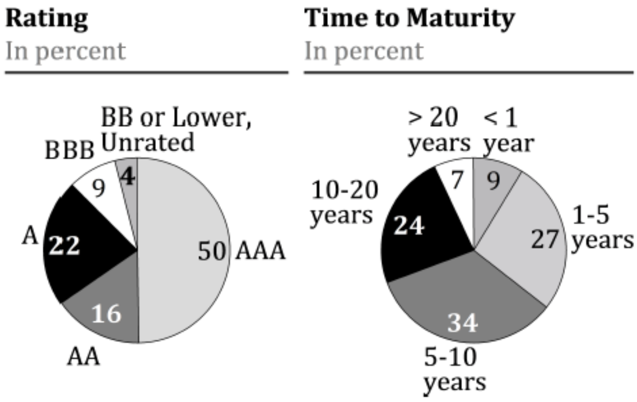
Source: Höring, 2012, p. 14.

service), insurance companies are not particularly interested in high returns, thus, the portfolio is invested relatively conservatively. The majority, 82% of the market risk portfolio is invested in debt, or fixed income instruments, 49% of which is invested into sovereign, supranational or agency debt. The remainder of the debt portion of the market risk portfolio is invested into corporate bonds and covered bonds, mainly mortgage covered bonds (Höring, 2012). Only 7% of the market risk portfolio is invested in equities and alternatives. The real estate properties make up 11% of the market risk portfolio and consist mainly of properties leased to third parties.

For the credit spread risk sub-module of the Solvency II standard approach, both credit risk of assets (determined by their credit ratings) and their time to maturity or modified duration are of importance. Therefore, we present an overview of the rating and the time to maturity distribution of the fixed income portfolio in Figure 2. 98% of sovereign debt instruments are rated investment grade, with the average rating of AA. Corporate debt is mostly investment grade, too, with 92.5%, and an average rating of A. (Covered bonds have an average rating of AAA due to the collateral backing.) Regarding the distribution of time to maturity, it is important to note that corporate bonds in the portfolio typically have short or medium-term maturities. For example, the average time to maturity of bank bonds is 5.6 years. An interesting and representative relationship of insurance investment portfolios is that the percentage of higher quality ratings increases with the time to maturity of fixed income instruments mainly due to the higher rating quality of longer-term sovereign bonds compared to short to medium-term corporate bonds. (Höring, 2012)

After having understood the basic business model of life insurance companies, we may try to

Figure 2: *Rating and time to maturity of the fixed income portfolio*



Source: Höring, 2012, p. 18.

collect the various types of risks inherent in their operation. The risk types are rather obvious in the asset side of the balance sheet. Equity and currency risk are straightforward to interpret. Property risk and intangible asset risk have a market risk part connected to the price changes of intangible assets and properties, and an internal risk part, inherent in the nature of these assets (failure). "Spread risk results from the sensitivity of the value of assets, liabilities and financial instruments to changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure." (EIOPA, 2014, p. 154) Interest rate risk is also connected to both the asset and the liability side: it is the risk associated with the revaluation of interest rate sensitive assets and liabilities resulting from a change in the term structure of interest rates. Counterparty default risk means the risk associated with the loss due to unexpected default of counterparties or debtors of the insurance undertaking, e.g. the default of the bank at which the insurance company has deposits. Finally, risk of market concentrations refers to the risk resulting from the uneven distribution of exposures among various debtors. (EIOPA, 2014)

In contrast, a bit of explanation is needed to understand the insurer's risks on the liability side, that is underwriting risks. First of all, insurers face moral hazard and adverse selection. Moral hazard means that after a policyholder has bought a contract, he or she does not fully bear the consequences of his or her actions. Adverse selection is a connotation of the fact that policyholders have more information about their own levels of risk than the insurance company. Thus, the insurer prices the contracts at an average level, so people with higher risk are more inclined to buy insurance contracts than people with lower risk. In reality, however, these risks are rarely responsible for an insurance company's failure. Rather, an insurer distress generally results from two separate causes that reinforce each other: corporate governance and external shocks (Lorent, 2008). Corporate governance is particularly important in case of insurance companies since, because the premium is fixed before knowing the cost of the claim, the final cost of insurance companies depends heavily on the determination of the premiums. Thus, since the final cost depends strongly on stochastic events, a random loss (e.g. a catastrophe) could severely deteriorate the insurer's capital even if the pricing was correct. However, in case of an underestimation of the value of liabilities, premiums cannot cover claims, therefore it can easily happen that the undertaking has to strive for higher returns by accepting higher risks. This excessive risk-taking creates room for an endogenous amplification of exogenous shocks. (Lorent, 2008)

Liquidity risk is typically ignored when analysing underwriting risks, since policyholders do not have the same rights as depositors of a bank. For example, policyholders often have to pay compensation for an early revocation or withdrawal of the contract. However, as Lorent (2008) argues, over the last years, the assimilation of banking-type activities by life insurers has made the industry more vulnerable to external shocks, especially to fluctuations in financial market conditions. This is particularly true in case of the so-called unit-linked contracts gradually gaining in popularity. "A unit-linked contract does not guarantee fixed cash payments, but instead a fixed number of shares in some investment vehicle." (Lorent, 2008, p. 8) Thus, the policyholder bears the investment risk. However, owing to the fierce competition, insurance companies introduced additional features to unit-linked products in the last couple of years, e.g. guaranteed returns, options to surrender the policy, etc... The surrender options, which are commonly included in these types of products, enable the policyholder to withdraw funds in case of interest rate changes or market fluctuations. Initially, there has been a surrender fee required to be paid in case of exercising this option, but, in the opinion of Lorent (2008), this fee is likely to vanish. "Consequently, policyholders will be more inclined to cancel their policies and change for other products and suppliers with no extra cost, increasing the volatility of business and reducing the duration of the liabilities for insurance companies." (Lorent, 2008, p. 9) Thus, among other problems[2], the liquidity of the liability side increases, introducing liquidity risk into the operation of insurance companies. Solvency II addresses this issue by phasing in a lapse risk in the computation of the SCR (see below). Lapse risk is related to the loss in the value of insurance liabilities from changes of the rates of policy lapses, terminations or surrenders (EIOPA, 2014; Lorent, 2008). This concludes the presentation about the various aspects of underwriting risk. Last but not least, however, we should also not forget operational risk from our list of risks of insurance undertakings, which results from inadequate internal processes, people and systems, or from external events (Eling, Schmeiser, & Schmit, 2007).

2.2 An overview of Solvency II

Although the current regulation (Solvency I) has provided an initial rules-based set of minimum capital requirements, those minimum capital requirements were not reflecting the risk structure borne by the insurance company, but only its business volume. Meanwhile, risk management

[2]Consider for example a flexible unit-linked contract with a guaranteed minimum return and a surrender option. In case of a decline in interest rates, the reduced income may not be enough to cover liabilities (with the guaranteed return). Conversely, if the interest rates rose, policyholders may be tempted to exercise their surrender options in order to invest in other products with higher returns (Lorent, 2008).

techniques have been developing together with market products introducing new risks. However, e.g. accounting for market or operational risks was nearly absent in Solvency I, in addition, assets were valued by book value instead of their market value. Therefore, in 2000, the European Commission launched the Solvency II project, a revision of the prudential regulation aiming to fully reflect the latest developments in the sector. (Eling et al., 2007; Lorent, 2008)

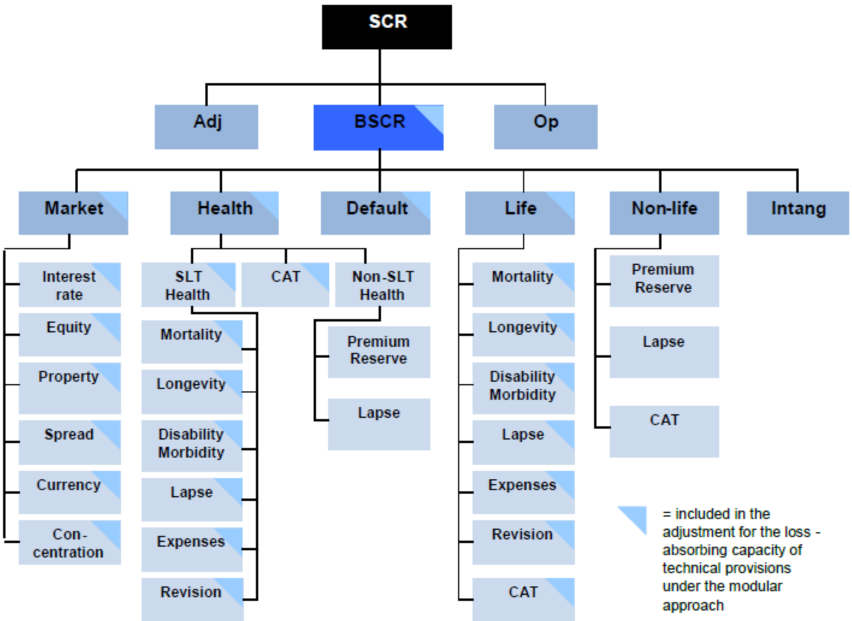
Solvency II is a new regulatory supervision framework for insurance companies[3] across the European Economic Area (EEA), introducing economic risk-based capital requirements. The project's implementation has started in 2003 using a Lamfalussy process, by which firstly the key principles were stipulated and ratified in a directive (Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance), then, the European Insurance and Occupational Pensions Authority (EIOPA) has received the task of elaborating the technical specifications (EIOPA, 2014). Solvency II is scheduled to come into effect on 1 January 2016, although the final implementation date has been postponed beforehand several times. (Höring, 2012)

"Solvency II is intended to establish a harmonised, principle-based, and risk-sensitive solvency supervision across the EEA based upon the actual risk profile of each individual insurance company to promote better regulation, comparability, transparency and international competitiveness, and to instil value-based management." (Höring, 2012, p. 7) Similar to Basel II, Solvency II will also consist of three pillars. Pillar 1 specifies the quantitative requirements. Two capital requirements are stipulated representing different levels of regulatory intervention. The *Solvency Capital Requirement* (SCR) is the target capital requirement, and is calibrated to the one-year VaR with 99.5% confidence level, which means that within the next year, the insurer's capital will only be insufficient to absorb unexpected losses in 0.5% of cases, or in 1 out of 200 cases. When a company goes below the SCR, it is required to provide additional reporting to the regulator including the reasons of the difficulties, as well as to disclose a financial recovery plan to restore the appropriate level of capital. In addition, there is also a *Minimum Capital Requirement* (MCR), calculated as a combination of a linear formula with a floor of 25% and a cap of 45% of the SCR, moreover, it is also subject to an absolute floor expressed in euros, depending on the nature of the undertaking (EIOPA, 2014). This is a trigger level under which the company should not go, otherwise supervisory authorities can take severe actions going from an intervention in the management actions to the closing to new business and the withdrawal of authorisation (Lorent, 2008). This double trigger system protects shareholders and the top management against a regulatory bias toward excessive interventionism (Lorent, 2008).

The calculation of the SCR is based on a holistic approach to the total balance sheet, which measures assets and liabilities at market values. The calculation covers all quantifiable risks, accounts for diversification, reinsurance and other risk-mitigation techniques, too, and takes also the loss-absorbing capacity of technical provisions and deferred taxes into account. To

[3]According to the directive, the regulation will apply to all insurers and reinsurers except the ones with annual gross premium income not exceeding €5 million, and pension funds.

Figure 3: *Overall structure of the SCR*



Source: EIOPA, 2014, p. 120.

calculate the SCR, insurers have the option to choose between five different methods (Gatzert & Martin, 2012): besides the standard formula provided by the regulator, which is calibrated on 'average' data across the European insurance industry, the SCR can be calculated by using the standard model with a partial internal model taking only a few risk parts into consideration. They are also allowed under certain conditions to use undertaking-specific parameters or simplifications. Finally, they may also choose to develop a full internal model for modelling all the insurer's risks. However, all internal models are subject to approval by supervisory bodies. The calculation structure of the SCR within the standard model is illustrated in Figure 3.

Pillar 2 comprises the qualitative elements of the new regulatory regime, such as a governance system ensuring to provide sound and prudent management, and specifies supervisory activities to provide early warning to the regulator and sufficient power for intervention (Höring, 2012; Lorent, 2008). The governance system covers risk management, internal control, the actuarial function, compliance, internal audit, as well as outsourced services, and is characterised by "a transparent organisation structure, a clear allocation and appropriate segregation of responsibilities, and an effective system for information transmission which is adequately documented and regularly reviewed." (Höring, 2012, p. 8) In addition, as part of their risk management system, all insurance undertakings should have a regular practice of assessing their overall solvency needs in an *Own Risk and Solvency Assessment* procedure (ORSA), to complement Pillar 1 requirements (Höring, 2012; Lorent, 2008). The ORSA is intended to address all material risks with a view of the company's specific risk profile, in a multi-year perspective. Finally, Pillar 2 also details the supervisory review process, a toolkit of supervisory activities to assess

the insurance company's ability to provide an early warning system for regulators. (Höring, 2012)
Pillar 3 is devoted to disclosure and reporting requirements to policyholders, investors, employees and supervisory authorities. In order to create transparency to enable market participants to exercise market discipline, the publication of a Solvency and Financial Condition Report is required. Moreover, provision of information to the supervisor through annual and quarterly quantitative reporting templates and the Regular Supervisory Report is obligatory, too.

2.3 Credit and market risk in the standard approach

As the partial internal model constructed later in this study accounts for credit and market risks, it is necessary not just to know the main features of Solvency II, but also to look more deeply into the calculation of modules of the standard formula in which credit and market risks are covered. Thus, our objective in this section is to present the methodology of the market risk module and the counterparty default risk module of the standard model in detail, based on the technical specifications (EIOPA, 2014).[4]
As we have already seen in Figure 3, there are two modules in which credit and market risks are involved: the market risk module deals with the risks resulting from changes in the level or volatility of market prices of financial instruments, whereas the counterparty default risk module should reflect possible losses due to unexpected default of the counterparties or debtors of the insurer over the forthcoming twelve months (EIOPA, 2014). The market risk module comprises six sub-modules, each dealing with interest rate, equity, property, spread and currency risk, and the risk arising from market concentrations, respectively. The capital requirements for the sub-modules are determined based on pre-specified scenarios, calibrated in a way that the resulting SCR should correspond to the VaR of Basic Own Funds (BOF)[5] of an insurance company subject to a 99.5% confidence level over a one-year period, as already mentioned in the previous section (EIOPA, 2014). The capital requirements for market sub-risks should be combined to an overall capital requirement for market risk using a predefined correlation matrix (the exact parameters are available in EIOPA, 2014, pp. 138-139).
The capital requirement of the *interest rate risk sub-module* accounts for the effects of increases and decreases in the term structure of interest rates on the value of technical provisions and assets sensitive to interest rate movements (fixed income investments, loans, interest rate derivatives). In the calculation it is assumed that the stresses of the scenario are only applied to the basic risk-free rates, thus any spread in excess of the risk-free return should remain unchanged in the stress scenarios. The capital requirement for interest rate risk is calculated as the biggest change in the net value of assets and liabilities due to the revaluation of

[4] We would like to warn the reader right at the start: the presentation in this section is only an overview underlining the key aspects of the calculations. It is not a fully detailed version of the regulation, thus, for exemptions and special references please consult the source.

[5] The level of Basic Own Funds is defined as the difference between assets and liabilities, where the liabilities does not contain subordinated liabilities.

all interest rate sensitive items using altered term structures which are assumed to be caused by an instantaneous stress. Specifically, the altered term structures are derived by multiplying the current interest rate curve by $(1 + s^{up}(t))$ or $(1 + s^{down}(t))$, where $s^{up}(t)$ and $s^{down}(t)$ mean the upward and downward stresses for individual maturities t specified in the technical specifications (EIOPA, 2014, pp. 142-143), respectively. However, there is an absolute minimum of one percentage point required for the increase of interest rates in the upward scenario at any maturity, irrespective of the stress factors. (EIOPA, 2014)

For the calculation of the capital requirement for *equity risk* in the standard approach, equities are split into two categories. Type 1 equities are listed in regulated markets in countries which are members of the EEA or the OECD, or are units or shares of collective investment undertakings established or marketed in the European Union. In contrast, type 2 equities are listed in stock exchanges in countries which are not members of the EEA or the OECD, equities which are not listed, hedge funds, commodities and other alternative investments (EIOPA, 2014). In addition, they shall also comprise investments other than those covered in the interest rate, property or spread risk sub-modules (EIOPA, 2014). One determines the capital requirements for type 1 and type 2 equities by applying an immediate 45.5% and 56.5% equity shock scenario, respectively, and then revaluing the net value of assets and liabilities. Afterwards, the capital requirement for equity risk is calculated by combining the capital requirements of the individual categories using a 75% correlation coefficient. The applied approach in the *property risk sub-module* is similar, but with a general instantaneous decrease of 25% in the value of investments in immovable property, which is defined as land, buildings and immovable-property rights, as well as property investments for the own use of the insurance company. (EIOPA, 2014)

For the determination of the capital requirement for *currency risk* arising from changes in the level or volatility of currency exchange rates, first the definition of the *local currency* is necessary. For the standard approach, local currency is the legal tender in which the enterprise prepares its financial statements. Therefore, for every relevant foreign currency, that is the currency in which the insurance company has a not hedged exposure, the capital requirement is determined as the maximum impact of two predefined scenarios: an instantaneous rise or an instantaneous fall of 25% in the value of the currency against the local currency. The total capital requirement for currency risk is then the sum over all currencies. (EIOPA, 2014)

The *spread risk* sub-module, which applies to corporate bonds, subordinated debt instruments, investment instruments with equity and bond features[6], covered bonds, as well as loans other than retail loans secured by a residential mortgage (residential mortgage loans are dealt with in the counterparty default risk module detailed below), and also securitisation positions and credit derivatives not for hedging purposes, accounts for the risk resulting from changes in the level or in the volatility of credit spreads over the risk-free term structure (EIOPA, 2014). The capital requirement for the spread risk sub-module (Mkt_{sp}) is calculated as the sum of the

[6] In case of assets exhibiting both fixed income and equity characteristics, one should determine which sub-module of the standard formula to apply with regard to the predominant economic form of the asset.

capital requirements for spread risk on bonds and loans other than residential mortgage loans (Mkt_{sp}^{bonds}), on securitisation positions ($Mkt_{sp}^{securitisation}$), and on credit derivatives (Mkt_{sp}^{cd}):

$$Mkt_{sp} = Mkt_{sp}^{bonds} + Mkt_{sp}^{securitisation} + Mkt_{sp}^{cd}.$$

The spread risk on bonds and loans other than residential mortgage loans is the immediate effect of the widening of their credit spreads on the net value of assets and liabilities. More precisely, this part of the capital requirement can be computed using

$$\sum_i MV_i \cdot F^{up}(rating_i, duration_i),$$

where the function $F^{up}(rating_i, duration_i)$ uses the credit quality step and the modified duration of the exposure, is capped at 100% and is calibrated to deliver a shock consistent with the 99.5% VaR requirement, and MV_i is the value of the credit risk exposure i. The specific parameterisation of the function is available in the technical specifications (EIOPA, 2014, p. 158)[7]. The calculation of the capital requirement for spread risk on securitisation positions uses the same logic as presented in the case of bonds, but splits securitisations into two categories. Type 1 securitisations are circumscribed in the technical specifications in a watchful manner, accounting for all features of securitisations with possible adverse implications. In contrast, for type 2 securitisations, which are only specified as the ones that do not qualify for the first category, much higher risk factors are used.

With regard to credit derivatives, two different scenarios are taken into consideration for the calculation of their capital requirement: an instantaneous absolute increase in credit spreads of the instruments underlying the credit derivative, as well as an instantaneous relative decrease in spreads of the instruments. The corresponding capital requirement will be derived using the type of shock that gives rise to the higher capital requirement.

Assets covered by the sub-module for *market risk concentrations* include assets considered in the equity, spread or property risk sub-modules, but assets on which the counterparty default risk module applies, do not belong to the scope of this sub-module. Concentration risk is defined as the risk of the accumulation of exposures with the same counterparty, hence e.g. the analysis of geographical concentrations is excluded from the standard formula. In order to perform the calculation, first we need to determine the relative excess exposure per single name exposure[8]. This is computed with the formula

$$XS_i = \max\left(0, \frac{E_i}{Assets} - CT\right),$$

where E_i are the single name exposure to counterparty i, $Assets$ is the total amount of assets considered in the calculation of this sub-module, and CT is the relative excess exposure

[7] The parameterisation presented on page 158 of the technical specifications is only a baseline version. There are, however, special references to covered bonds, exposures to central governments and central banks, multilateral development banks and international organisations. In particular, a risk factor of 0% should apply to central governments and central banks of all EU member states, which raises a number of problems, as Gatzert and Martin (2012) point out.

[8] Groups or conglomerates are dealt with under one single name exposure.

threshold, depending on the credit quality of the exposure. This relative excess exposure is then multiplied by a parameter also depending on the credit quality step of the counterparty[9], to result in the capital requirement for market risk concentration on a single name exposure. The capital requirement for this sub-module is computed assuming no correlation among the requirements for each counterparty:

$$Mkt_{conc} = \sqrt{\sum_i (Conc_i^2)}.$$

The *counterparty default risk module* is created to reflect possible losses from an unexpected default of counterparties or debtors of the insurance company within the following year. Contracts included in this module are risk-mitigating contracts (reinsurance arrangements, securitisations and derivatives), receivables from intermediaries and any other credit exposure not covered in the spread risk sub-module. Within the module, two kinds of exposures are determined according to their characteristics, and different treatments are applied to them. The common characteristic of type 1 exposures is that they may not be diversified and the counterparties of these exposures are likely to be rated, such like risk-mitigation contracts, cash at bank, commitments received by the company which have been called up but are unpaid, etc... Conversely, the class of type 2 exposures covers the ones which are usually diversified and the counterparty of which is probably unrated, e.g. receivables from intermediaries, policyholder debtors, residential mortgage loans, etc...

The capital requirements of each category ($SCR_{def,1}$ and $SCR_{def,2}$) should be calculated separately. In the aggregation, a low diversification effect should be allowed:

$$SCR_{def} = \sqrt{SCR_{def,1}^2 + 1.5 \cdot SCR_{def,1} \cdot SCR_{def,2} + SCR_{def,2}^2}.$$

The calculation of the capital requirement for type 2 exposures is quite simple. It is determined as the loss in the BOF that would result from an instantaneous decrease in value by $0.9 \cdot LGD_{receivables>3months} + 0.15 \cdot \sum_i LGD_i$, where $LGD_{receivables>3months}$ denotes the losses-given-default on all receivables from intermediaries due for more than three months, and the sum is taken on all the losses-given-default of type 2 exposures other than these.

In contrast, type 1 exposures do not have a simple capital requirement calculation. After having determined the estimated loss-given-default (LGD) and probability of default (PD) of the exposures in the type 1 portfolio, the capital requirement is calculated as follows:

$$SCR_{def,1} = \begin{cases} 3 \cdot \sqrt{V}, & \text{if} \quad \sqrt{V} \leq 7\% \cdot \sum_i LGD_i \\ 5 \cdot \sqrt{V}, & \text{if } 7\% \cdot \sum_i LGD_i < \sqrt{V} \leq 20\% \cdot \sum_i LGD_i \\ \sum_i LGD_i, & \text{if} \quad 20\% \cdot \sum_i LGD_i \leq \sqrt{V}, \end{cases}$$

where V is the variance of the loss distribution for type 1 exposures, which is the sum of V_{inter} and V_{intra}, which are determined as

$$V_{inter} = \sum_{(j,k)} \frac{PD_k \cdot (1 - PD_k) \cdot PD_j \cdot (1 - PD_j)}{1.25 \cdot (PD_k + PD_j) - PD_k \cdot PD_j} \cdot TLGD_j \cdot TLGD_k$$

[9]The thresholds and parameters depending on the credit quality categories are specified in the technical specifications (EIOPA, 2014, pp. 170-171).

and

$$V_{intra} = \sum_j \frac{1.5 \cdot PD_j \cdot (1 - PD_j)}{2.5 - PD_j} \cdot \sum_{PD_j} LGD_i^2,$$

where $TLGD_j$ and $TLGD_k$ denote the sum of losses-given-default on type 1 exposures from counterparties bearing a probability of default PD_j and PD_k, respectively. The probabilities of default depend on credit quality steps where credit ratings are available.

The standard formula also provides estimates for the losses-given-default, i.e. the loss of BOF which the insurer would occur if the counterparty defaulted. The LGD is determined differently for different types of contracts. For a reinsurance arrangement or a securitisation,

$$LGD = \max(0; 50\%(Recoverables + 50\%RM) - F \cdot Collateral),$$

where $Recoverables$ is the best estimate of recoverables from the reinsurance contract, RM is the risk-mitigating effect of the reinsurance arrangement on underwriting risk, $Collateral$ is the risk-adjusted value of collateral in relation to the reinsurance arrangement and F is a factor taking into account the economic effect of having a collateral in relation to the reinsurance arrangement or securitisation in case of any credit event related to the counterparty in question. For a derivative contract,

$$LGD = \max(0; 90\%(MarketValue + 50\%RM) - F \cdot Collateral),$$

where $MarketValue$ is the value of the derivative, RM is the risk-mitigating effect of the derivative on market risk, and everything else is defined as previously. In contrast, the loss-given-default for a mortgage loan should be calculated as

$$LGD = \max(0; Loan - Mortgage),$$

with $Mortgage$ being the risk-adjusted value of a mortgage, that is the difference between the value of the residential property held as mortgage and the adjustment for market risk, and $Loan$ being the value of the mortgage loan.

Although the formulas and the main paths of the calculations presented above for both the market risk and the counterparty default risk module relevant for us may seem a bit complicated, the attentive reader has probably noticed that these computations do not require any modelling: provided that credit ratings and some simple measures about each of the exposures of the asset portfolio are available, closed-form formulas are at service. Trivially, these formulas are calibrated to an average European insurer and do not distinguish between insurance companies with different risk characteristics. Therefore, formulas in the standard approach are parameterised rather conservatively, thus, with an internal model tailed on its specific risks, an insurance company is able to significantly reduce its solvency capital requirement. But how is it possible to develop a model capturing all the relevant types of risks discussed above? Hopefully, after the introduction of the most important approaches to credit risk in the next section, we will be able to provide an answer. However, when developing an internal model, one should not overlook the fundamentals of the Solvency II regulation, but devise his or her own approach comprising these characteristics. These fundamentals in my opinion are: the scenario approach calibrated to correspond to the one-year 99.5% VaR, the holistic approach which deals with all relevant and quantifiable risks, and the dependencies whose precise grasp is of key importance.

3 Credit risk models

As discussed in the previous section, insurance companies are motivated by the conservative specification of the Solvency II standard approach to develop internal models which appropriately address risks inherent in their operation, since, in general, this reduces their solvency capital requirement. Thus, when referring to the narrower topic of this article, credit risk, there is also motivation for modelling as many types of risks considered relevant by the regulation as possible. In addition, as emphasized by Jarrow and Turnbull (2000) and Crouhy, Galai, and Mark (2000), although the regulations usually deal with credit risk and market risk separately, "market risk and credit risk are intrinsically related to each other and [...] are not separable. If the market value of the firm's assets unexpectedly changes – generating market risk – this affects the probability of default – generating credit risk. Conversely, if the probability of default unexpectedly changes – generating credit risk – this affects the value of the firm – generating market risk." (Jarrow & Turnbull, 2000, p. 272) Furthermore, the Solvency II regulation mentions spread risk as an element of the market risk module. However, as Crouhy et al. (2000) point out, fluctuations in credit spreads may appear either because equilibrium conditions in capital markets change, because the credit quality of some obligors has improved or deteriorated or the simultaneous occurrence of both (Crouhy et al., 2000). Moreover, "disentangling market risk and credit risk driven components in spread changes is further obscured by the fact that often market participants anticipate forthcoming credit events before they actually happen. Therefore, spreads already reflect the new credit status when the rating agencies effectively downgrade an obligor, or put him on 'credit watch'." (Crouhy et al., 2000, p. 60) Thus, when modelling credit risk, we are trying to consider and incorporate all sorts of risks affecting the asset side of insurers. That is, we are looking for a model which can properly capture the specific types of credit risk insurers face, and which is flexible enough to allow for the inclusion of various other risk factors related to credit risk, e.g. interest rate risk, exchange rate risk, and so on. To do this, in this section we delineate the most important credit risk models developed in the literature and provide a comparative assessment.

Before dwelling on the discussion of various credit risk models, however, we would like to underline some major difficulties regarding credit risk modelling. The first is the relative scarcity of data: we typically not only lack the information regarding a firm's true credit quality and economic prospects, but there is also a shortage in credit events which hinders sound statistical inference. This is aggravated by the fact that in credit risk management, the usual time horizon is one year, on the basis of the time necessary to liquidate a portfolio of credit-risky instruments.

This is in sharp contrast with the typical time horizon of market risk management, which calls for different approaches. Secondly, a typical credit loss distribution differs significantly from the normal distribution: it is strongly skewed with a heavy upper tail. That is, over the years, small profits are rather frequent, accompanied by occasional large losses. And lastly, default dependence has a crucial impact on the upper tail of the credit loss distribution for a portfolio, thus, precise capturing of the dependence structure is of key importance. Obviously it is not plausible to expect independence among exposures. Since different firms are affected by common macroeconomic factors, we have dependence between their defaults. Moreover, default dependence can also be caused by direct economic links between firms, such as a strong borrower-lender relationship, although this is probably less prominent when considering large portfolios. (Jarrow, & Turnbull, 2000; McNeil, Frey, & Embrechts, 2005)
One can divide credit risk models into two main categories: structural and reduced-form models. In a structural (or otherwise known as firm-value or threshold) model, default occurs when a stochastic variable, or in dynamic approaches, a stochastic process typically representing the asset value of a firm falls below a threshold representing liabilities (McNeil et al., 2005). In contrast, in a reduced-form model, we do not specify the price mechanism leading to default: the default time of a firm is modelled as a non-negative random variable, whose distribution typically depends on economic covariables (McNeil et al., 2005). We introduce structural models first since their way of thinking is more easily understandable and constitutes the base of modern credit risk modelling.

3.1 Structural models

The Merton model

We are starting our discussion with the Merton model, which is the progenitor of all firm-value models, and is still an influential benchmark for academics as well as a popular setup for practitioners. We discuss the model based on McNeil et al. (2005). Consider a firm whose asset value follows a stochastic process (V_t). The firm finances itself by equity S_t and debt, which, taken as simply as possible, consists of one zero-coupon bond with face value B and maturity T. This means that for any $0 \leq t \leq T$, the balance sheet equality is represented by: $V_t = S_t + B_t$. It is assumed that the firm cannot pay out any dividends or issue new debt. Default can only occur at the maturity of the bond, T, since that is the timing of the firm's only payment obligation. If at maturity $V_T > B$, there is no default: the bondholders will receive B, and the shareholders the remaining $S_T = V_T - B$. If, on the other hand, $V_T \leq B$, the firm cannot meet its financial obligations, thus the debtholders liquidate the firm and share the proceeds among themselves: $V_T = B_T$. Moreover, $S_T = 0$ since the shareholders receive nothing, but thanks to their limited liability, they also do not pay anything. Thus, the value of the firm's equity at time T equals the payoff of a European call option on the firm's asset value V_T (McNeil et al., 2005):

$$S_T = \max(V_T - B, 0) = (V_T - B)^+.$$

If we further assume that the asset value process follows a geometric Brownian motion under the real-world probability measure P:

$$\mathrm{d}V_t = \mu_V V_t \,\mathrm{d}t + \sigma_V V_t \,\mathrm{d}W_t,$$

where $\mu_V \in \mathbb{R}, \sigma_V > 0$ and W_t is a standard Wiener process, the well-known Black-Scholes pricing formula is easily computable:

$$S_t = C^{BS}(t, V_t, r, \sigma_V, B, T) = V_t \Phi(d_{t,1}) - Be^{-r(T-t)}\Phi(d_{t,2}), \quad \text{where}$$

$$d_{t,1} = \frac{\ln V_t - \ln B + (r + \frac{\sigma_V^2}{2})(T-t)}{\sigma_V \sqrt{T-t}} \quad \text{and} \quad d_{t,2} = d_{t,1} - \sigma_V \sqrt{T-t}.$$

The firm's debt is also possible to be priced similarly, since its value is the face value of the liabilities minus the payoff of a European put option on V_T with strike price B. Note that we implicitly assumed a normal distribution of the asset returns, and also that the pricing formula uses the risk-neutral probability measure, which from now on we will denote by Q. In order to calculate the true default probability of our firm, we need to return to the real-world probability measure (McNeil et al., 2005):

$$P(V_T \leq B) = P(\ln V_T \leq \ln B) = \Phi\left(\frac{\ln(B/V_0) - (\mu_V - \frac{\sigma_V^2}{2})T}{\sigma_V \sqrt{T}}\right).$$

Credit migration models and CreditMetrics

A different type of structural models are the so-called *credit migration models*. In a credit migration model, each firm is assigned to a credit rating category at any given time point. There are a finite number of such ratings, they are ordered by credit quality and include the category of default. Rating agencies, such as Moody's or Standard & Poor's (S&P) frequently publish credit ratings for major companies and sovereigns, and also the so-called rating transition matrices, which determine the probability of moving from one credit rating to another credit rating over the given risk horizon (e.g. one year). These transition matrices are estimated from historical default data. An example of a transition matrix is given in Figure 4. (McNeil et al., 2005)

In this framework we assume that the current credit rating completely determines the default probability, thus this probability can be read off from the transition matrix. For larger time horizons, cumulative default probabilities are also estimated directly and published by rating agencies, such as Figure 5 by Standard & Poor's. The relationship between the one-year transition matrix and the average cumulative default rates is not trivial. A usual assumption is to consider the credit migration process as a time-homogeneous Markov chain. In this case, the n-year transition matrix is simply the n-fold product of the one-year transition matrix, so the n-year default probabilities can be read off from the last column of the n-year transition matrix. However, the assumption of the Markov property has been criticized heavily in the literature.(McNeil et al., 2005)

Figure 4: *An example of a one-year rating transition matrix*

Initial rating	Rating at year-end (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0.00	0.00	0.00
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0.00
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	1.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0.00	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0.00	0.22	1.30	2.38	11.24	64.86	19.79

Source: Standard & Poor's CreditWeek (15 April 1996) (as cited in McNeil et al., 2005, p. 339).

It is important to note as well that empirical default rates tend to vary with the state of the economy, being high during recessions and low during expansion periods. However, transition rates estimated by rating agencies are historical averages over longer time horizons covering several business cycles ("through-the-cycle" estimates). This is consistent with the fact that rating agencies focus on the average credit quality of a firm when determining its credit rating. Thus, in case we are rather interested in "point-in-time" estimates of default probabilities reflecting the current macroeconomic environment, we have to adjust the provided historical transition matrices, for example by incorporating information about Treasury rates or an equity index. (McNeil et al., 2005)

The CreditMetrics modelling framework was developed by J. P. Morgan and is one of the most influential industry standards. The framework is best summarized by Figure 6. The calculation of "Value at Risk due to Credit" is based largely upon a rating system and the transition matrices estimated accordingly. Both transition matrices and default probabilities published by rating agencies and internal estimates by financial institutions can be applied. A strong assumption made by CreditMetrics is that all issuers are credit-homogeneous within the same rating class, with same transition probabilities and the same default probability. (This assumption is relaxed in the KMV model detailed below, where each firm has a specific default probability.) Nevertheless, using this assumption, and with the help of the transition matrix appropriate for our risk horizon, we can now determine the probability of migration into any rating for any rating class (thus, any issuer). The next step towards the one-year forward distribution of the bond value consists of specifying the forward zero curve at the risk horizon for each credit category, and, in case of default, the value of the instrument which is usually set at a percentage, named "recovery rate" of face value. The forward price of the bond in e.g. one year from now is derived from the forward zero curve one year ahead, which is then applied to the residual cash flows from year one to the maturity of the bond. The recovery rates for different seniority classes are estimated from historical data by the rating agencies.[10] (Crouhy et al., 2000)

[10]In the Monte Carlo simulation used to generate the loss distribution, it is assumed that the recovery rates are

Figure 5: *Average cumulative default rates*

Initial rating	Term							
	1	2	3	4	5	7	10	15
AAA	0.00	0.00	0.07	0.15	0.24	0.66	1.40	1.40
AA	0.00	0.02	0.12	0.25	0.43	0.89	1.29	1.48
A	0.06	0.16	0.27	0.44	0.67	1.12	2.17	3.00
BBB	0.18	0.44	0.72	1.27	1.78	2.99	4.34	4.70
BB	1.06	3.48	6.12	8.68	10.97	14.46	17.73	19.91
B	5.20	11.00	15.95	19.40	21.88	25.14	29.02	30.65
CCC	19.79	26.92	31.63	35.97	40.15	42.64	45.10	45.10

Source: Standard & Poor's CreditWeek (15 April 1996) (as cited in McNeil et al., 2005, p. 340).

Using these calculations, we now are able to derive the forward distribution of the changes in the bond value. The distribution exhibits long downside tails (an illustrative example is given by Crouhy et al. (2000) in Figure 7), thus its credit-VaR will be much larger than computed with assuming a normal distribution for the changes in bond value. (Crouhy et al., 2000)

In order to determine the credit-VaR of a portfolio of loans and bonds[11], we have to specify the dependence structure of our instruments, that is their joint default and migration probabilities. This is the point where CreditMetrics inevitably adjoins to the logic of other structural models: it connects the default correlation of issuers to their asset correlation. To illustrate how this is done, let us first consider only one company. Having the transition matrix, we know the probability of default for each of our bonds. Then, we can easily assign a threshold according to the Merton model for which the probability of default with respect to the real-world probability measure is exactly the probability of default read off from the transition matrix:

$$P(V_T \leq d_1) = \bar{p},$$

where d_1 denotes the default threshold and $\bar{p}$ the default probability observed in the transition matrix. In the same vein, we are able to translate any migration probabilities to rating thresholds, that is we can choose thresholds

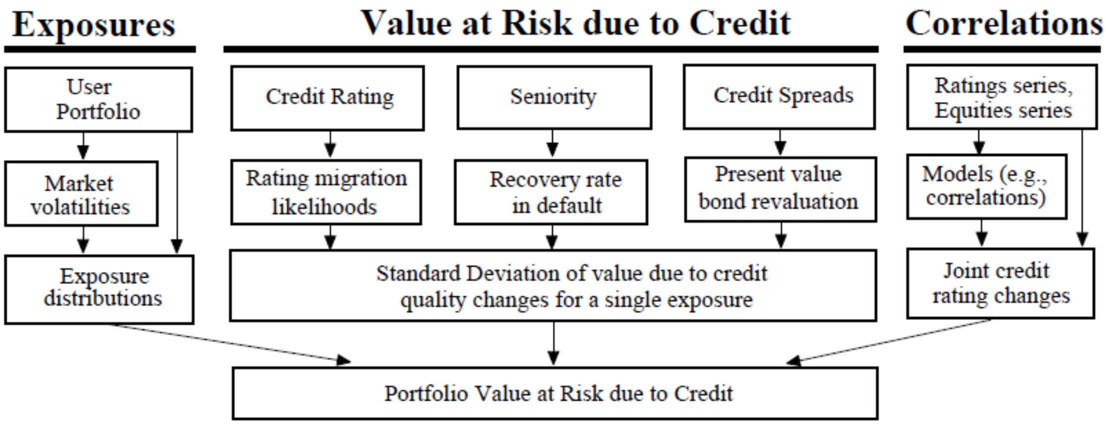
$$-\infty = d_0 < d_1 < \cdots < d_n < d_{n+1} = +\infty$$

such that $P(d_j < V_T \leq d_{j+1}) = \bar{p_j}$ for $j \in \{0, \ldots, n\}$ (McNeil et al., 2005). Thus, turning now to a portfolio consisting of m instruments, we can connect the joint default and migration probabilities to the joint distribution of the asset values. Let us only show the case of two states: default or non-default. Let $Y_i := Y_{T,i}$ be a default indicator variable for firm i so that $Y_i \in \{0, 1\}$

distributed according to a beta distribution with the mean and standard deviation estimated from historical data for each seniority class.

[11] The CreditMetrics approach also allows the incorporation of various other instruments, such as receivables, loan commitments, financial letters of credit and derivatives.

Figure 6: *The CreditMetrics framework*



Source: Gupton, Finger, & Bhatia, 1997, pp. iv.

and $Y_i = 1 \Leftrightarrow V_{T,i} \leq d_{1,i}$, and $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ the vector of default indicators for the portfolio and $p(\boldsymbol{y}) = P(Y_1 = y_1, \ldots, Y_m = y_m), \boldsymbol{y} \in \{0,1\}^m$ its joint probability function. Then:

$$P(Y_1 = 1, \ldots, Y_m = 1) = P(V_{T,1} \leq d_{1,1}, \ldots, V_{T,m} \leq d_{1,m})$$

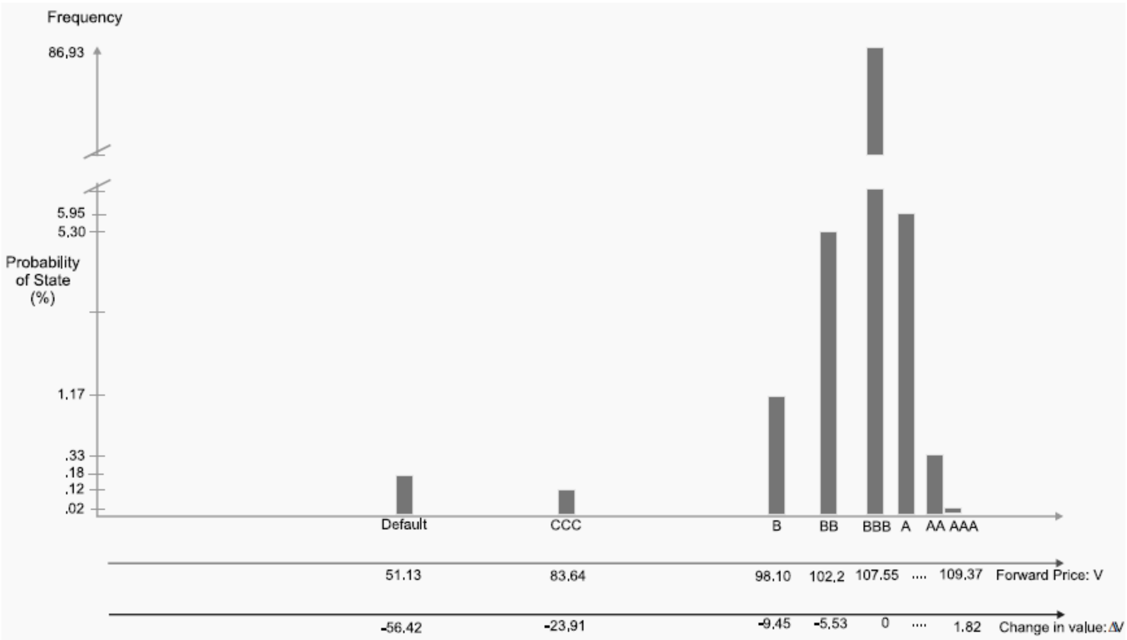
The joint distribution, when various states of credit ratings are allowed for, can be calculated similarly. The reason for establishing this connection was that the direct estimation of the correlation between default indicators is hardly possible. However, by specifying the asset value processes as geometric Brownian motions, we already implicitly assumed that they have normal marginal distributions at any time point. If we further assume their joint distribution to be of multidimensional normal, we only have to estimate the covariance matrix.[12] Nevertheless, since asset values are still not directly observable, equity prices for publicly traded firms are used as a proxy to calculate asset correlations. For a large portfolio of bonds and loans, this would still require the computation of a huge correlation matrix. To reduce the dimensionality of the estimation problem, CreditMetrics uses a factor model. This approach maps each obligor to the countries and industries that most likely determine its performance. In this way, equity returns are correlated to the extent that they are exposed to the same industries and countries. More formally, the vector of critical variables with standard normal marginal distributions (which is transformed from the multidimensional normal asset returns) $\boldsymbol{V}$ can be written as

$$\boldsymbol{V} = B\boldsymbol{F} + \varepsilon$$

for a p-dimensional random vector of common factors $\boldsymbol{F} \sim N_p(\boldsymbol{0}, \Omega)$ with $p < m$, a loading matrix $B \in \mathbb{R}^{m \times p}$, and an m-dimensional vector of independent univariate normally distributed

[12] Since it is well-known that a (multidimensional) normal distribution can be unequivocally determined by its mean and covariance matrix. The joint normality, however, is not a straightforward assumption, it means we are working with a Gauss copula connecting marginal and joint distributions. Models with alternative copula structures (e.g. a t-copula) are also worth considering.

Figure 7: *An example histogram of the 1-year forward prices and changes in the value of a BBB-rated bond*



Source: Crouhy et al., 2000, p. 71.

errors ε, which are also independent from $\boldsymbol{F}$. Here, the random vector $\boldsymbol{F}$ represents country and industry effects. This factor structure implies that the covariance matrix Σ of $\boldsymbol{V}$, which, since we assumed a multidimensional standard normal vector, will be a correlation matrix, is of the form $\Sigma = B\Omega B' + \Upsilon$, where Υ is the diagonal covariance matrix of ε. (Crouhy et al., 2000; Gupton et al., 1997; McNeil et al., 2005)

The KMV model

As Crouhy et al. (2000) point out, the accuracy of the CreditMetrics methodology relies upon two strong assumptions: all firms with the same credit rating have equal default rates and these actual default rates correspond to the historical average default rate. That is, the ratings and the default rates are synonymous and the rating changes if and only if the default rate is adjusted. Obviously, this raises a number of problems, since default rates are continuous and rating agencies need time to track changes in companies' default risk. In addition, KMV has demonstrated that substantial differences may exist within the same rating classes, and the overlap in default probability ranges may be quite large. (Crouhy et al., 2000)
Therefore, KMV does not use any rating agency's statistical data. Instead of assigning default probabilities to obligors based only on their credit rating category, they derive actual default probabilities called Expected Default Frequencies (EDFs) with a logic similar to the Merton model. The EDFs are firm-specific, but they can also be easily mapped into any rating system to derive the equivalent rating of the obligor. The KMV model thus does not make an explicit ref-

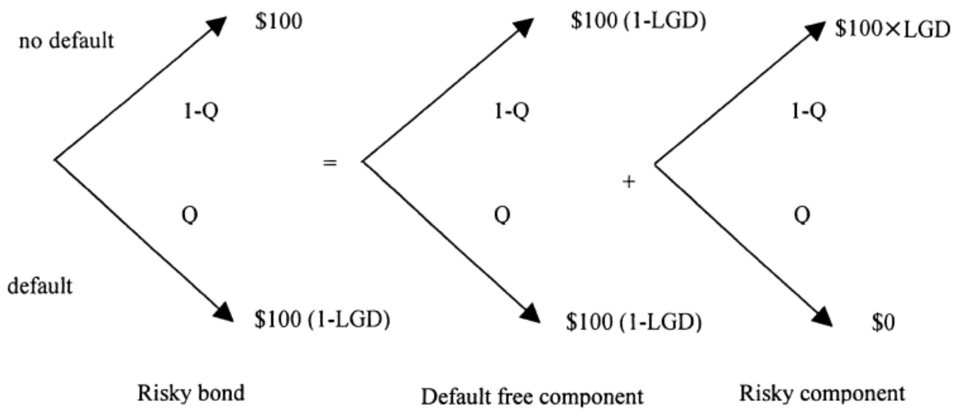
erence to the transition probabilities, since credit ratings are embedded into the EDFs. (Crouhy et al., 2000)
The derivation of the EDFs in the KMV framework (which we will introduce based on Crouhy et al. (2000)) proceeds in three steps: estimation of the market value and volatility of the firm's assets; calculation of the distance-to-default, which is an index measure of default risk; and mapping the distance-to-default to actual default probabilities using a default database. (Crouhy et al., 2000)
Usually, the market value of the firm's assets is assumed to be lognormally distributed, that is the log asset returns follow a normal distribution. According to KMV, this assumption is quite robust when challenged on empirical grounds, and the asset distribution is rather stable over time (which means its volatility is almost constant). However, the asset value process is not directly observable, thus we need to resort to an estimation based on Merton's logic. Under some simplifying assumptions about the firm's liability structure, it is possible to derive analytical solutions for the value of equity V_E and its volatility, σ_E:

$$V_E = f(V_A, \sigma_A, K, c, r) \qquad \sigma_E = g(V_A, \sigma_A, K, c, r),$$

where K denotes the leverage ratio in the capital structure, c is the average coupon paid on long-term debt and r is the risk-free interest rate. If σ_E was observable or at least relatively stable over time, we could estimate it from the equity value process and solve the above system of equations for V_A and σ_A. Since this is not the case, thus there is no simple way to measure σ_E precisely from market data, KMV solves the transformed equation $V_A = h(V_E, \sigma_A, K, c, r)$ by an iterative technique. (Crouhy et al., 2000)
"In the option pricing default, or equivalently, bankruptcy, occurs when assets value falls below the value of the liabilities. In practice, default is distinct from bankruptcy which corresponds to the situation where the firm is liquidated, and the proceeds from assets sale is distributed to the various claim holders according to pre-specified priority rules. Default is the event when a firm misses a payment on a coupon and/or the reimbursement of principal at debt maturity. [...] KMV has observed from a sample of several hundred companies that firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt. Therefore, the tail of the distribution of asset value below total debt value may not be an accurate measure of the actual probability of default." (Crouhy et al., 2000, pp. 88-89) For this reason KMV does not compute the probability of default directly from asset values but proposes an index called distance-to-default (DD), which is the number of standard deviations between the mean of the asset value distribution and the "default point", set at the par value of current liabilities, including short-term debt to be serviced over the time horizon and half the long-term debt ($DPT = STD + 1/2LTD$). Formally, given our earlier assumptions about the asset value process:

$$DD = \frac{\ln(\frac{V_0}{DPT_T}) + (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

Figure 8: *Valuation of a single cash flow subject to default risk*



Source: Crouhy et al., 2000, p. 98.

where V_0 is the current market value of assets, DPT_T the default point at time horizon T, μ the expected net return on assets, and σ the annualized asset volatility.
Following this, the last phase consists of mapping the DD to the actual default probabilities called EDFs, for a given time horizon. This is accomplished by using historical information based on a large sample of firms, since here, the use of the Merton model's logic would produce considerably lower default rates than actually observed.
In contrast to the bond (cash flow) valuation methodology of CreditMetrics (forward zero curves specified to each rating category), KMV uses a rather different approach called risk-neutral valuation or martingale approach, where we price securities as the discounted expected value of future cash flows. The expectation here is calculated under the risk-neutral probability measure, which is computable according to the following formula:[13]

$$Q_T = N\left[N^{-1}(EDF) + \frac{\mu - r}{\sigma}\sqrt{T}\right].$$

Let us illustrate the risk-neutral valuation first on a single cash flow (a zero-coupon bond)! Assume we know the risk-free term structure of interest rates and we have an estimation for the loss given default (LGD). Then, a single risky cash flow element is valued with the recognition that it can be divided into a risk-free and a risky cash flow component (illustrated in Figure 8). Both components can be discounted with the risk-free interest rate since we are under the martingale measure. Thus:

$$PV = 100(1 - LGD)e^{-r_T T} + [100LGD(1 - Q_T) + 0Q_T]e^{-r_T T}.$$

Knowing that if we had not had risk in our cash flow, we would have calculated the price according to $PV = 100e^{-r_T T}$, we may now determine the credit spread (CS) for the risky cash

[13]This formula is only precise if $EDF = N(-DD)$. However, this is not exactly the case, thus in pratice, an estimation is required. There is a detailed discussion about this estimation in Crouhy et al. (2000).

flow as the solution of

$$100(1-LGD)e^{-r_T T}+[100LGD(1-Q_T)+0Q_T]e^{-r_T T}=100e^{-(r_T+CS_T)T}.$$

Generalizing the pricing method for a stream of cash flows $[C_1,\ldots,C_i,\ldots,C_n]$:

$$PV=(1-LGD)\sum_{i=1}^{n}C_ie^{-r_it_i}+LGD\sum_{i=1}^{n}(1-Q_i)C_ie^{-r_it_i},$$

where Q_i denotes the cumulative risk-neutral EDF at the horizon t_i. This concludes the KMV model since the calculation of default correlations follows exactly the methodology presented in the previous section for CreditMetrics. (Crouhy et al., 2000)

When we want to compare the KMV model and a model based on credit migrations (such as CreditMetrics), we can lay down the following advantages of the KMV model. First, the EDF reacts quickly to changes in the economic prospects of the firm, as these tend to be reflected in the firm's share price and hence in the estimated DD (McNeil et al., 2005). Secondly, the EDFs tend to reflect the current macroeconomic environment, rising in recessions and decreasing in times of prosperity. However, as for drawbacks of the KMV method, it is quite sensitive to global over- and underreaction of equity markets, which can be dangerous if a KMV-type model is used to determine the regulatory capital of a financial institution. Lastly, and rather obviously, the KMV model can be only applied to firms with publicly traded stock, whereas a credit migration model is available for any firm rated by some (maybe internal) rating system (McNeil et al., 2005).

There are some disadvantages worth mentioning for all structural models discussed above. First of all, many inputs of the models cannot be directly observed, hence, estimation techniques have to be used the accuracy of which is rather difficult to check. Second, interest rates are assumed to be deterministic (Jarrow, & Turnbull, 2000). Third, as pointed out by Jarrow and Turnbull (2000), as the maturity of a credit-risky bond tends to zero, the credit spread also tends to zero, which feature is not supported by empirical observations. Thus, let us discuss a different modelling methodology which might have solutions for the deficiencies of the previous models.

3.2 Reduced-form models

CreditRisk$^+$

CreditRisk$^+$, an industry model for credit risk proposed by Credit Suisse First Boston (1997), applies actuarial techniques to the derivation of the loss distribution of a portfolio of credit-risky exposures. Only default risk is modelled, not downgrade risk. In CreditRisk$^+$, just like in other reduced-form models, no assumption is made about the causes of default: an obligor is either in default or not, with a certain probability. The distributional assumptions and functional forms imposed by CreditRisk$^+$ allow the distribution of total portfolio losses to be calculated in a convenient analytic form. Default correlations are assumed to be driven entirely by a vector of risk factors $\mathbf{\Psi}$. Conditional on $\mathbf{\Psi}$, defaults of individual obligors are assumed to be independently distributed Bernoulli draws. The conditional probability of default $p_i(\mathbf{\Psi})$ for obligor i is a function of the rating grade $\zeta(i)$ of obligor i, the realization of risk factors $\mathbf{\Psi}$, and a vector of factor loadings $\boldsymbol{w_i} = (w_{i,1}, \ldots, w_{i,k})$, which measure the sensitivity of obligor i to each of the risk factors ($\sum_{j=1}^{k} w_{i,j} = 1$):

$$p_i(\mathbf{\Psi}) = \bar{p}_{\zeta(i)} \sum_{j=1}^{k} \psi_j w_{i,j}.$$

We first aim to derive the conditional probability generating function[14] $F(z|\,\boldsymbol{\psi})$ for the total number of defaults in the portfolio given realization $\boldsymbol{\psi}$ of the risk factors. For a single obligor i, this is the probability generating function of a Bernoulli random variable:

$$F_i(z|\,\boldsymbol{\psi}) = (1 - p_i(\boldsymbol{\psi}) + p_i(\boldsymbol{\psi})z) = (1 + p_i(\boldsymbol{\psi})(z-1)).$$

Using the approximation formula $\ln(1+y) \approx y$ for $y \approx 0$, we can write

$$F_i(z|\,\boldsymbol{\psi}) = \exp(\ln(1 + p_i(\boldsymbol{\psi})(z-1))) \approx \exp(p_i(\boldsymbol{\psi})(z-1)).$$

We call this step the Poisson approximation, since the expression on the right-hand side is the probability generating function of a Poisson distributed random variable. "The intuition is that, as long as $p_i(\boldsymbol{\psi})$ is small, we can ignore the constraint that a single obligor can default only once, and represent its default event as a Poisson random variable rather than as a Bernoulli. The exponential form of the Poisson probability generating function is essential to the computational facility of the model." (Gordy, 2000, p. 122)

Conditional on $\boldsymbol{\psi}$, default events are independent across obligors, so the probability generating function for the total number of defaults is the product of the individual probability generating functions:

$$F(z|\,\boldsymbol{\psi}) = \prod_i F_i(z|\,\boldsymbol{\psi}) \approx \prod_i \exp(p_i(\boldsymbol{\psi})(z-1)) = \exp(\mu(\boldsymbol{\psi})(z-1)),$$

where $\mu(\boldsymbol{\psi}) \equiv \sum_i p_i(\boldsymbol{\psi})$. To get the unconditional probability generating function, we need to integrate out the $\boldsymbol{\psi}$, that is:

$$F(z) = \int_{\mathbf{\Psi}} F(z|\,\boldsymbol{\psi})\,\mathrm{d}H(\boldsymbol{\psi}),$$

where $H(\boldsymbol{\psi})$ is the cumulative distribution function of the factor vector $\mathbf{\Psi}$. To specify this, the risk factors in CreditRisk+ are assumed to be independent gamma-distributed random variables with mean one and variance σ_j^2, $j = 1, \ldots, k$. By integrating out, it can be shown that the resulting probability generating function will be of a form of a probability generating function of a sum of k independent negative binomial variables. (Gordy, 2000)

[14]The probability generating function of a discrete random variable X taking values in the non-negative integers is defined as: $F(z) = E(z^X) = \sum_{x=0}^{\infty} p(x)z^x$.

After calculating the distribution for the number of defaults[15], the next phase is to derive the loss distribution. Let us assume that loss given default is a constant fraction λ of the exposure, which is L_i for obligor i. (The exposures are exogenous to the model.) In order to retain the computational advantages of the discrete model, we need to express the loss exposure amounts λL_i as integer multiples of a fixed unit of loss v_0 (e.g. one million dollars). These integer multiples are called *standardized exposure* levels. The standardized exposure for obligor i denoted $v(i)$ is equal to $\lambda L_i/v_0$ rounded to the nearest integer. (Gordy, 2000)

Let us now calculate the probability generating function of losses! The probability of a loss of $v(i)$ units of a portfolio consisting only of obligor i must equal the probability that i defaults, thus $G_i(z|\,\boldsymbol{\psi}) = F_i(z^{v(i)}\big|\,\boldsymbol{\psi})$. Using the conditional independence of defaults, the conditional probability generating function is

$$G(z|\,\boldsymbol{\psi}) = \prod_i G_i(z|\,\boldsymbol{\psi}) = \exp\left(\sum_{j=1}^{k}\psi_j \sum_i \bar{p}_{\zeta(i)} w_{i,j}(z^{v(i)}-1)\right).$$

It is again possible to integrate out ψ in order to arrive at the unconditional probability generating function of losses. (Gordy, 2000)

CreditRisk⁺ presents the advantage of being relatively easy to implement: due to closed-form expressions, it is computationally attractive and requires relatively few inputs to estimates. However, the methodology assumes no market risk, and it also ignores migration risk: the exposures are fixed and do not depend on changes in the credit quality of issuers, as well as on the variability of future interest rates. (Crouhy et al., 2000)

Intensity models

Intensity-based credit risk modelling goes back to the seminal papers of Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Das and Tufano (1995) and Duffie and Singleton (1999). In this section I am going to present the most important features of intensity models based on McNeil et al. (2005), Giesecke (2002) and Lando (2004).

In reduced-form models, as we have already seen in the context of CreditRisk⁺, we do not try to explain defaults. Instead, the dynamics of default are prescribed exogenously: default occurs without warning, at an exogenous default rate, or intensity. In order to understand why this is an advantageous specification, let us present the main building blocks of an intensity model. We consider default as a random time τ, which is an $\mathscr{F}$-measurable random variable on the probability space $(\Omega, \mathscr{F}, P)$, taking values in $[0, \infty]$. Let its cumulative distribution function, which represents the probability of default until a certain time point t, be denoted by $F(t) = P(\tau \leq t)$. Then, we call $\bar{F}(t) = 1 - F(t)$ the survival function of τ. Moreover, the function $\Gamma(t) :=$

[15] Although we only determined the probability generating function, this uniquely identifies the distribution. In addition, the cumulative distribution function can be derived in the following way:

$$p(k) = P(X = k) = \frac{F^{(k)}(0)}{k!}.$$

$-\ln(\bar{F}(t))$ is called the *cumulative hazard function* of τ. In case F is absolutely continuous with density f, the function $\gamma(t) := \frac{f(t)}{\bar{F}(t)}$ is called the *hazard rate* of τ. (McNeil et al., 2005)

From the definition $\Gamma(t) = \int_0^t \gamma(s)\,\mathrm{d}s$ since $\Gamma'(t) = \frac{f(t)}{\bar{F}(t)} = \gamma(t)$. The hazard rate can be interpreted as the instantaneous default probability at t given the survival up to t. To see this, let $h > 0$ and consider the probability $P(\tau \leq t+h \,|\, \tau > t) = \frac{F(t+h)-F(t)}{\bar{F}(t)}$. From this we obtain

$$\lim_{h\to 0} \frac{1}{h} P(\tau \leq t+h \,|\, \tau > t) = \frac{1}{\bar{F}(t)} \lim_{h\to 0} \frac{F(t+h)-F(t)}{h} = \gamma(t),$$

which underlines our interpretation. To define the default intensity, let us first introduce the *default indicator process* as a point process

$$Y(t) = 1_{\{\tau \leq t\}} = \begin{cases} 1 & \text{if } \tau \leq t \\ 0 & \text{else,} \end{cases}$$

as well as its natural filtration $\mathscr{H}_t = \sigma(\{Y_u : u \leq t\})$. Then, a proposition states (McNeil et al., 2005, pp. 394-395) that with the above given random time τ with default indicator process Y_t, absolutely continuous distribution function F and hazard rate function γ, $M(t) := Y(t) - \int_0^{\tau\wedge t} \gamma(s)\,\mathrm{d}s$ is an $(\mathscr{H}_t)$-martingale. This means that $\int_0^{\tau\wedge t} \gamma(s)\,\mathrm{d}s = \int_0^t \gamma(s) 1_{\{\tau > s\}}\,\mathrm{d}s$ is the compensator of the upward-trending default indicator process. Based on the above stated proposition, the hazard rate γ is also known as the *martingale intensity process* of the random time τ. This is what we call a default intensity process in reduced-form credit risk models and is usually denoted by λ. It is now straightforward to see that its interpretation is the same as of the hazard rates. (Giesecke, 2002; McNeil et al., 2005)

This λ intensity process is explicitly modeled in intensity models and calibrated from market data. For instance, the process is often assumed to be stochastic. Under this assumption, the random time τ is said to follow a *doubly stochastic Poisson process* otherwise known as a *Cox process*. For the exact definition, let us first define the background filtration $\mathscr{F}_t$, which symbolizes the information contained in the stochastic structure of the intensity process, as well as $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t$, which is all the information available to investors. Then, McNeil et al. (2005, p. 397) defines a doubly stochastic random time with respect to the background filtration $\mathscr{F}_t$ as a random time τ that admits the $(\mathscr{F}_t)$-conditional hazard rate process γ if $\Gamma(t) = \int_0^t \gamma(s)\,\mathrm{d}s$ is strictly increasing and if for all $t > 0$,

$$P(\tau \leq t \,|\, \mathscr{F}_\infty) = P(\tau \leq t \,|\, \mathscr{F}_t).$$

This last equation can be most easily interpreted if we assume that $\mathscr{F}_t$ is generated by some stochastic state variable processes. Then, the equation states that, given past values of the processes, the future does not contain any extra information for predicting the probability that τ occurs before time t (McNeil et al., 2005). The proposition defining the martingale intensity process can be extended to doubly stochastic random times, too. (McNeil et al., 2005)

The simulation of doubly stochastic random times is quite straightforward, based on the following lemma (McNeil et al., 2005, p. 398).

Lemma 3.1. *Let E be a standard exponentially distributed random variable on $(\Omega, \mathscr{F}, P)$ independent of $\mathscr{F}_\infty$, that is $P(E \leq t \,|\, \mathscr{F}_\infty) = 1 - e^{-t}$ for all $t \geq 0$. Let (γ_t) be a positive $(\mathscr{F}_t)$-adapted process such that $\Gamma(t) = \int_0^t \gamma(s)\,\mathrm{d}s$ is strictly increasing and finite for every $t > 0$. Define a random time τ by*

$$\tau := \Gamma^{-1}(E) = \inf\{t \geq 0 : \Gamma_t \geq E\}.$$

Then τ is doubly stochastic with $(\mathscr{F}_t)$-conditional hazard rate process γ.

Thus, given the realization of the intensity process γ, the simulation of default times can be easily accomplished based on this result. What about the pricing of a zero-coupon bond? What pricing formulas can we derive for credit risky bonds in this framework?

Let us consider a firm whose default time is a doubly stochastic random time τ on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), Q)$, where the economic background filtration $(\mathscr{F}_t)$ represents the information generated by an arbitrage-free and complete model for non-defaultable security prices (McNeil et al., 2005, p. 415), and Q is already the equivalent martingale measure which, as is well-known from the basic theorems of mathematical finance, is needed for pricing instead of the physical measure P. Furthermore, let us denote the default-free short interest rate by r_t. Hence, the default-free savings account is defined by $B_t = \exp(\int_0^t r_s\,\mathrm{d}s)$.

In the above discussed framework, we claim that the price of a risky bond is possible to be constructed from the following two basic building blocks: a *vulnerable claim* and a *recovery payment*. A vulnerable claim is an $\mathscr{F}_T$-measurable promised payment X made at time T if default did not occur. The actual payment is $X1_{\{\tau > T\}}$. In contrast, a recovery payment at time τ of default is $Z_\tau 1_{\{\tau \leq T\}}$, where $Z = (Z_t), t \geq 0$ is an $(\mathscr{F}_t)$-adapted stochastic process and $Z_\tau = Z_{\tau(\omega)}(\omega)$.

It is well-known that the price of an arbitrary $\mathscr{F}_T$-measurable claim X can be expressed as

$$X_t = B_t E_Q\left(\frac{X}{B_T}\middle|\mathscr{G}_t\right) = E_Q\left(\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)X\middle|\mathscr{G}_t\right)$$

Obviously, if our claim is default-free, the additional information about the default history contained in $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t$ is not relevant for computing the conditional expectation:

$$E_Q\left(\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)X\middle|\mathscr{G}_t\right) = E_Q\left(\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)X\middle|\mathscr{F}_t\right).$$

Lando (1998) was the first to show that in a similar vein, the pricing of the above introduced building blocks can be simplified to a pricing problem in a default-free security market model with adjusted default-free short rates (Lando, 1998, pp. 104-106; McNeil et al., 2005, pp. 417-418). (I omit the proofs but they are detailed in the two cited works.)

Theorem 3.2. *Let τ be doubly stochastic with background filtration $(\mathscr{F}_t)$ and hazard rate process (γ_t) under the equivalent martingale measure Q. Define $R_s := r_s + \gamma_s$. Assume that $\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)|X|$ and $\int_t^T |Z_s\gamma_s| \exp\left(-\int_t^s R_u\,\mathrm{d}u\right)\mathrm{d}s$ are all integrable with respect to Q. Then:*

$$E_Q\left(\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)1_{\{\tau>T\}}X\middle|\mathscr{G}_t\right) = 1_{\{\tau>t\}}E_Q\left(\exp\left(-\int_t^T R_s\,\mathrm{d}s\right)X\middle|\mathscr{F}_t\right),$$

and

$$E_Q\left(\exp\left(-\int_t^\tau r_s\,\mathrm{d}s\right)Z_\tau\,\middle|\,\mathscr{G}_t\right) = 1_{\{\tau>t\}}E_Q\left(\int_t^T Z_s\gamma_s\exp\left(-\int_t^s R_u\,\mathrm{d}u\right)\mathrm{d}s\,\middle|\,\mathscr{F}_t\right).$$

This means that for the calculation of the pricing formulas in intensity models, more specifically, for obtaining closed-form expressions for the expected values in Theorem 3.2, the entire machinery of default-free term structure modelling will be at our service. This is an enormous advantage of intensity models.

To further specify the pricing formulas, let us discuss the various recovery assumptions prominent in the literature. Firstly, Jarrow and Turnbull (1995) proposed the *recovery of treasury* (RT) assumption. Under RT, in case of default at time τ, the owner of the defaulted bond receives ϕ_τ units of the default-free zero-coupon bond at time τ, where ϕ_τ denotes the recovery rate. Under this recovery assumption and additionally assuming that the recovery rate ϕ_τ is an exogenously given constant ϕ, Jarrow and Turnbull (1995) show that the price of a risky zero-coupon bond can be expressed by the formula

$$v(t,T) = E_Q\left(\frac{B_t}{B_T}(\phi 1_{\{\tau\le T\}} + 1_{\{\tau>T\}})\,\middle|\,\mathscr{F}_t\right).$$

Moreover, if we also assume that the short rate process and the default intensity process are statistically independent under the equivalent martingale measure Q[16], the above equation further simplifies to

$$\begin{aligned} v(t,T) = E_Q\left(\frac{B_t}{B_T}\,\middle|\,\mathscr{F}_t\right)E_Q\left(\phi 1_{\{\tau\le T\}} + 1_{\{\tau>T\}}\,\middle|\,\mathscr{F}_t\right) &= \\ = p(t,T)\left(\phi + (1-\phi)Q(\tau > T\,|\,\mathscr{F}_t)\right)&, \end{aligned}$$

that is, the risky zero-coupon bond's price can be determined by knowing the default-free zero-coupon bond's value and the distribution of the default time under the martingale measure. (Jarrow, & Turnbull, 1995)

Since in reality, recovery is a complex issue with many legal and institutional features, all the recovery models presented here can only be a crude approximation of reality. However, as debt with the same seniority is assigned the same fractional recovery regardless of maturity, the *recovery of face value* (RF) assumption comes closest to legal practice. This assumption states that if default occurs at time τ, the bondholder gets immediately at τ a recovery payment of size ϕ_τ. Thus, the value at maturity T is given by

$$v(T,T) = 1_{\{\tau>T\}} + \frac{\phi_\tau}{p(\tau,T)}1_{\{\tau\le T\}}.$$

Since, even with strong simplifying assumptions, the recovery payment at maturity is dependent on the exact timing of default, the pricing of recovery payments under RF is rather involved. (McNeil et al., 2005)

[16] The authors mention that the reasonableness of this assumption is an outstanding empirical question which awaits empirical testing of the restricted model, and is only assumed for simplicity of implementation. Das and Tufano (1995) and Lando (1998) relax this assumption.

The recovery assumption that probably leads to the simplest pricing formulas has been put forward by Duffie and Singleton (1999). Under the *recovery of market value* (RM) assumption, the recovery payment at the time of default will be the fraction ϕ_τ of the value of the bond right before default. Notice that this is a recursive definition as the pre-default bond price depends on the recovery payment. Nevertheless, it has an economic meaning since this is the loss in value associated with default. Lando (2004) provides three different arguments to support this assumption, of which we only present a simple interpretation through the notion of thinning. (This is also presented in Lando (1998).) From a pricing perspective, the RM assumption is extremely convenient as it permits to use only the vulnerable claim part of a zero-coupon bond even with a nonzero recovery rate, albeit with a small adjustment in the intensity process. This is because receiving a fraction ϕ_τ of pre-default value in the event of default is equivalent to receiving the full pre-default value with probability ϕ_τ and getting zero with probability $1-\phi_\tau$. That is, we may view our default process with recovery rate ϕ_τ as a default process with zero recovery but with a thinned default intensity: $\gamma(1-\phi_\tau)$ instead of γ. With this consideration, it is now obvious that a claim with recovery rate ϕ_τ and recovery assumption RM has a pricing formula (Lando, 1998, 2004)

$$v(t,T) = 1_{\{\tau>t\}} E_Q \left(\exp\left(-\int_t^T r_s + \gamma_s(1-\phi_s)\,\mathrm{d}s \right) X \,\middle|\, \mathscr{F}_t \right).$$

So far, we have only been busy with building up the stochastic structure of the model and with pricing a zero-coupon bond. However, we would also like to examine the credit risk of a portfolio. In order to do that, we also need to make assumptions regarding the dependence structure of the portfolio. Intensity models are very flexible in this question, too. The simplest construction is the assumption of conditional independence among defaults. That is, given the realization of the background processes governing the default intensity, defaults are independent. Notice that this is the dynamic extension of the structure in CreditRisk+. For a more precise definition, let us consider m obligors with random default times $\tau_i, i = 1, \dots m$ on the probability space $(\Omega, \mathscr{F}, P)$ and with the usual background filtration $\mathscr{F}_t$. Then, according to McNeil et al. (2005, p. 431), the $\tau_1, \dots \tau_m$ default times are conditionally independent doubly stochastic random times if they are doubly stochastic random times (which we have already defined earlier) and they are conditionally independent given $\mathscr{F}_\infty$, that is for $t_1, \dots t_m$, we have

$$P(\tau_1 \le t_1, \dots \tau_m \le t_m \,|\, \mathscr{F}_\infty) = \prod_{i=1}^m P(\tau_i \le t_i \,|\, \mathscr{F}_\infty).$$

The simplest algorithm for simulating conditionally independent doubly stochastic random times is the obvious extension of the simulation of a doubly stochastic random time based on Lemma 3.1: now we need to generate independent standard exponentially distributed random draws. However, more efficient algorithms for simulation are also available, see McNeil et al. (2005). An important advantage of the assumption of conditional independence is that the pricing formulas of single-name credit products (that is products whose payoff depends only on the default history of a single firm) obtained in the single-firm version of the model remain valid in the portfolio version as well. To see this, define $\mathscr{H}_t^i$ as the default information filtration of firm i and $\mathscr{H}_t$ as the default information filtration of all firms in the portfolio, as well as $\mathscr{G}_t^i = \mathscr{H}_t^i \vee \mathscr{F}_t$ and $\mathscr{G}_t = \mathscr{H}_t \vee \mathscr{F}_t$ as the information available to investors regarding a single firm i and all firms in the portfolio, respectively. Then, for the price of an arbitrary, $\mathscr{F}_T$-measurable vulnerable claim of the form $H = 1_{\{\tau_i > T\}}X$, it is obviously true because of conditional independence that

$$E_Q\left(\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)H\middle|\mathscr{G}_t^i\right) = E_Q\left(\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)H\middle|\mathscr{G}_t\right),$$

which is generally not valid assuming not conditionally independent structures. (McNeil et al., 2005)

Let us show a simple but popular example with the assumption of conditional independence! (This example is discussed both in Giesecke (2002) and in McNeil et al. (2005).) Here, the intensity process is modelled as a linear combination of independent affine diffusions:

$$\gamma_{t,i} = \gamma_{i0} + \sum_{j=1}^{p} \gamma_{ij}\psi_{t,j}^{syst} + \psi_{t,i}^{id}, \qquad i = 1, \dots m, \tag{3.2.1}$$

where $(\psi_{t,j}^{syst})$, $1 \leq j \leq p$ and $(\psi_{t,i}^{id})$, $i = 1, \dots m$ are independent affine diffusions representing common or systematic factors and idiosyncratic factors, respectively. Here, the dependence among the defaults of firms comes from the sensitivity of their intensities on common factors, which means that when the uncertainty about the realization of the common factors is removed, defaults occur independently of each other. In the literature, both CIR-type square-root diffusion processes, and, more generally, basic affine jump diffusion processes have been used to model the dynamics of the factors driving the intensity. (Giesecke, 2002; McNeil et al., 2005)

Despite the tractability of the assumption of conditional independence among defaults, one may wants to use more sophisticated models to better grasp for example the high tail dependence among defaults, or to explicitly model default contagion and counterparty risk. Then, one needs to resort to copula models or models with interacting intensities. For the definition of copula models recall that we can generate conditionally independent doubly stochastic random times with $(\mathscr{F}_t)$-adapted intensity processes $(\gamma_{t,1}), \dots (\gamma_{t,m})$ by finding a random vector $\mathbf{E}$ of independent, unit exponentially distributed components independent of $\mathscr{F}_\infty$ and taking $\tau_i = \inf\{t \geq 0 : \Gamma_{t,i} \geq E_i\}$, or, slightly differently written

$$\tau_i = \inf\{t \geq 0 : 1 - \exp(-\Gamma_{t,i}) \geq U_i := 1 - \exp(-E_i)\},$$

where $\mathbf{U} = (U_1, \dots U_m)'$ is a vector with m independent, uniformly distributed margins, whose joint distribution function gives us the m-dimensional independence copula. In general, however, we are free to define copulas other than the independence copula, too, which provides us an opportunity for constructing rich dependence structures. On the other hand, if we wanted to model counterparty risk explicitly, we may write the default intensity of a firm i in the portfolio as a function of the current state $\mathbf{Y_t}$ of the portfolio besides common factors $\boldsymbol{\psi_t}$: $\gamma_{t,i}(\boldsymbol{\psi_t}, \mathbf{Y_t})$. This way we can explicitly put counterparty risk into the model since, if for instance firm j has an important trading or borrower-lender relationship with firm i, then we are now able to explicitly specify the influence of the default of firm i on the default intensity of firm j, in this case with a positive coefficient of Y_i in the intensity process γ_j. These precise specifications, however, can easily lead to a model with very involved calculations. (Giesecke, 2002; McNeil et al., 2005)

After the main features of intensity models have been reviewed, I would like to shortly get back to the first question which may arise in the reader: is it not a problem that we do not explicitly try to explain the occurrence of default in reduced-form models? Well, for the case of CreditMetrics and CreditRisk$^+$, Gordy (1998) provides a mapping between the two approaches. Furthermore, as Lando (2004) argues, intensity models are fully consistent with a structural approach in models where bondholders have incomplete information about the asset value of the firm. Thus, if we take the role of incomplete information in structural models into account, we can reach a common perspective on the structural and reduced-form approaches for analysing credit risk. The detailed discussion of this approach is, however, beyond the scope of this paper. (Giesecke, 2002; Gordy, 1998; Lando, 2004)

4 The model and its implementation

"A value-at-risk measure that successfully integrates market, credit and liquidity risk is the Holy Grail of a successful risk management procedure." (Jarrow, 2001, p. 1) With this end in view, albeit – given that liquidity risk is not yet a prominent risk in the business model of a life insurance company – setting liquidity risk aside, in this section we are going to try to build a credit risk model that also incorporates some aspects of market risk, under the requirements of Solvency II. This model can give an example of a partial internal model whose usage by insurers – as a complement to the standard method – is allowed according to the Solvency II regulation. Furthermore, if it is successful at grasping the specific credit and market risks a certain insurance undertaking is bearing, it will probably be able to arrive at smaller solvency capital requirements than the rather conservatively calibrated standard method exhibited in section 1.

In order to meet the requirements of Solvency II, let us recall the most important aspects the regulation expects from partial internal models, emphasized at the end of section 1! Solvency II requires the models to determine the one-year value-at-risk (VaR) at 99.5% confidence level. In addition, the regulation represents a holistic approach towards the risks an insurance undertaking is exposed to, thus, possibly all types of risks should be accounted for in the model. Now it is important to mention that the partial internal model presented in this section aims only to address the risks inherent in the fixed income portfolio of a life insurance company. Hence, in contrast to the standard method, this model will not address equity risk, counterparty default risk or property risk. Incorporating equity risk into the framework is possible: one may proceed the way Gatzert and Martin (2012) suggested. For taking counterparty default risk into account, besides working with a bigger portfolio than the fixed income portfolio and incorporating e.g.

reinsurance arrangements, securitisations, derivatives, cash at bank, debt from policyholders, residential mortgage loans, etc..., we would need a highly complicated dependence structure in the model, such as a reduced-form model with interacting intensities, which would have made the model too complicated to be tractable, that is why we refrained from such a specification. Regarding property risk, its market risk part could have been grasped by adding immovable properties on the asset side of the insurer's balance sheet to our portfolio. However, its other part (failure) could have been addressed by actuarial methods, which is not the topic of this study. This also implies that we are not going to provide a full solvency capital requirement, since it is defined as the VaR of basic own funds which is in essence the difference between assets and liabilities. Thus, in order to come up with an SCR, we need to determine the VaR of liabilities, too, which involves calculation methods for underwriting risk, also not the aim of this paper.

In summary, this paper sets itself a target of providing a partial internal model for the risks of the fixed income portfolio of a life insurance undertaking, including interest rate risk, currency risk and credit spread risk, and experimenting with combining all these types of risks in a single model. In addition, credit ratings have been an important input for the standard method, recognizing the fact that taking rating grades into account can significantly improve the risk assessment of a particular bond. Hence, we are also going to incorporate credit ratings into the model.

In order to choose the appropriate model for our requirements, we also need to be aware of other characteristics. Firstly, sovereign bonds make up typically almost half of our fixed income portfolio. However, sovereign bonds are rather difficult to handle with a structural model as structural models (recall e.g. CreditMetrics or the KMV model) build heavily upon the asset price process or upon the share price process of companies. These are obviously not at our disposal in case of sovereigns. Hence, if we want to take sovereign bonds into account, we have to resort to reduced-form models.

A second feature we want our model to be capable of is to include a flexible dependence structure among exposures. Moreover, for determining a part of the solvency capital requirement of an insurance company, we cannot make use of a "through-the-cycle" estimate of default or transition rates since we are not interested in the probability of default or migration in a typical year but instead in the actual year ahead of us. Thus, it is of high importance to us to adjust the default and migration rates according to the business cycle, in this way arriving at "point-in-time" estimates. This is fortunately not a binding criterion for reduced-form models. However, trying to account for interest rate risk properly, we would like the default-free term structure to be stochastic, which is rather problematic in the CreditRisk$^+$ framework.

Hence, we resort to an intensity model which, although much more demanding from a theoretical point of view, offers high flexibility and allows to introduce all the desired features listed above. In particular, the calibration procedure does not involve share prices but only bond price data, which is by definition available for all the exposures in our portfolio. Furthermore, being dynamic models, intensity models also support the inclusion of various market risks, e.g. a stochastic default-free term structure. It is also possible to incorporate credit ratings into an intensity model as the seminal work of Jarrow, Lando and Turnbull (1997) has demonstrated. However, their paper provides only a pricing framework, and applies strong simplifying assumptions regarding the statistical independence of the default-free term structure and default intensities under the martingale measure. In order to allow for a flexible dependence structure as well as point-in-time default and migration probabilities, we introduce stochastic hazard rates depending upon some state variables in the model of Jarrow et al. (1997), following the work of Lando (1998).

Based on Lando (1998), we exhibit a general model in the next subsection, which could be easily of use as a partial internal model for insurance companies. In the second subsection, we will assume a simplified structure and implement the model on real data.

4.1 Lando (1998)'s model and some extensions

In the model of Jarrow et al. (1997), the distribution for default time is modelled via a continuous time, time-homogeneous Markov chain on a finite state space $S = \{1, \dots K\}$. These states represent the various rating classes, with 1 being the highest and $K-1$ being the lowest grade. State K is the default state. In contrast to the transition matrices we have already seen when discussing CreditMetrics, in continuous time the time-homogeneous Markov chain $\{\eta_t : 0 \leq t \leq \tau\}$ can be specified with its $K \times K$ generator matrix

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_{12} & \lambda_{13} & \dots & \lambda_{1K} \\ \lambda_{21} & -\lambda_2 & \lambda_{23} & \dots & \lambda_{2K} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \dots & -\lambda_{K-1} & \lambda_{K-1,K} \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

where $\lambda_{ij} \geq 0$ for $i \neq j$, and $\lambda_i = \sum_{j=1, j\neq i}^{K} \lambda_{ij}$, $i = 1, \dots K-1$. The off-diagonal terms of the transition matrix represent the transition rates of jumping from rating i to j. Consequently, e.g. $\lambda_1 \Delta t$ is the probability that a firm in rating class 1 will jump to a different class or default within a small time interval Δt. Default is an absorbing state as implied by the last row of zeros. The t-period transition matrix for η is given by: (Jarrow et al., 1997)

$$P(t) = \exp(t\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(t\mathbf{A})^k}{k!}.$$

Let us now generalize the transition matrix to include stochastic transition intensities depending upon state variables, according to Lando (1998)! Let our generator matrix be of a form

$$\mathbf{A_X(s)} = \begin{pmatrix} -\lambda_1(X_s) & \lambda_{12}(X_s) & \lambda_{13}(X_s) & \dots & \lambda_{1K}(X_s) \\ \lambda_{21}(X_s) & -\lambda_2(X_s) & \lambda_{23}(X_s) & \dots & \lambda_{2K}(X_s) \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \lambda_{K-1,1}(X_s) & \lambda_{K-1,2}(X_s) & \dots & -\lambda_{K-1}(X_s) & \lambda_{K-1,K}(X_s) \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

with $\lambda_i(X_s) = \sum_{j=1,j\neq i}^{K} \lambda_{ij}(X_s)$, $i = 1, \dots K-1$. If these transition matrices (which are now stochastic) are at our service given a realization of the vector process of state variables X_s, we are capable of performing transition simulations with the model in the following way.

Let $E_{11}, \dots E_{1K}, E_{21}, \dots E_{2K}, \dots$ be a sequence of independent, unit exponentially distributed random variables and let a firm's initial rating class be η_0. With the help of Lemma 3.1, define

$$\tau_{\eta_0,i} = \inf\left\{t : \int_0^t \lambda_{\eta_0,i}(X_s)\,\mathrm{d}s \geq E_{1i}\right\}, \qquad i = 1, \dots K$$

and let $\tau_{\eta_0} = \min_{i\neq\eta_0} \tau_{\eta_0,i}$. That is, the firm's credit rating will jump to a class whose cumulative stochastic hazard function firstly hits its unit exponentially distributed threshold. Notice that when $\lambda_{\eta_0,i}(X_s)$ is large for some i, the cumulative hazard function grows faster and reaches the exponential threshold level faster. Therefore the probability of $\tau_{\eta_0,i}$ becoming smaller is higher, which means a jump to the rating class i will become more probable. The new rating class may be defined as $\eta_1 = \arg\min_{i\neq\eta_0} \tau_{\eta_0,i}$. If this is class K, the firm defaults and the simulation stops. Otherwise let

$$\tau_{\eta_1,i} = \inf\left\{t : \int_{\tau_{\eta_0}}^t \lambda_{\eta_1,i}(X_s)\,\mathrm{d}s \geq E_{2i}\right\}, \qquad i = 1, \dots K$$

and $\tau_{\eta_1} = \min_{i\neq\eta_1} \tau_{\eta_1,i}$ and also $\eta_2 = \arg\min_{i\neq\eta_1} \tau_{\eta_1,i}$ and so forth, either until default or until a certain end time T. We denote the time of default by τ: $\tau = \inf\{t : \eta_t = K\}$. With this construction, Lando (1998) claims that we can obtain a continuous time process which, given the realization of the state variables is a non-homogeneous Markov chain. (Lando, 1998)

How is it possible to price defaultable bonds in this model? Lando (1998) provides a general pricing formula for zero-coupon bonds with zero recovery for simplicity in the following theorem. (The proof is omitted for readability but is available in Lando (1998, p. 110).)

Theorem 4.1. *Let a zero-coupon bond maturing at time T be issued by a firm whose initial credit rating is i and its transition intensities evolve according to η. Then, assuming zero recovery, the price of the bond is*

$$v^i(t,T) = E_Q\left(\exp\left(-\int_t^T R(X_s)\,\mathrm{d}s\right)(1 - P_X(t,T)_{i,K}\,\middle|\,\mathscr{F}_t\right), \tag{4.1.1}$$

where $\mathscr{F}_t$ is the usual background filtration generated by the state variables, Q is the martingale measure, $R(X_s)$ is the default-free short rate allowed to depend upon the state variables, and $P_X(t,T)_{i,K}$ is the (i,K)-th element of the transition matrix for the period (t,T), constructed from the generator matrix process $A_X(s)$.

This is a general specification. However, for our purposes, we would like a model more restricted in its assumptions but also more tractable and computationally attractive. With this end in view, Lando (1998) recommends restricting the structure of the transition matrix according to the following lemma. (Again, we omit the proof but refer to Lando (1998, p. 111) for its description.)

Lemma 4.2. *Let $\mu_i : \mathbb{R}^d \to [0, \infty)$ be a non-negative function defined on the d-dimensional state space X for $i = 1, \dots K-1$ and assume that $\int_0^T \mu_i(X_s)\,\mathrm{d}s$ is finite almost surely. Moreover, let $\mu(X_s)$ denote the $K \times K$ diagonal matrix constructed as diag$(\mu_1(X_s), \dots, \mu_{K-1}(X_s), 0)$ and assume the time-dependent generator matrix has a representation*

$$A_X(s) = B\mu(X_s)B^{-1}, \tag{4.1.2}$$

where B is the matrix comprising the K eigenvectors of $A_X(s)$.[17] *Now, if we define*

$$E_X(s,t) = \begin{pmatrix} \exp\left(\int_s^t \mu_1(X_u)\,\mathrm{d}u\right) & 0 & \dots & 0 \\ 0 & \ddots & \dots & 0 \\ \vdots & \dots & \exp\left(\int_s^t \mu_{K-1}(X_u)\,\mathrm{d}u\right) & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

then with $P_X(s,t) = BE_X(s,t)B^{-1}$, $P_X(s,t)$ becomes the transition matrix of an inhomogeneous Markov chain on $\{1, \dots K\}$.

With the restrictions imposed in Lemma 4.2, the following bond pricing formula can be obtained (Lando, 1998).

Theorem 4.3. *Assume that conditionally on X, rating transitions are driven by the generator matrix $A_X(s)$ whose structure is defined according to Lemma 4.2. Moreover, let $R(X)$ denote the default-free short rate process, possibly depending upon the state variables. Then, the time t price of a zero-coupon bond maturing at T, issued by a firm initially in rating class i with zero recovery rate is given by*

$$v^i(t,T) = \sum_{j=1}^{K-1} \beta_{ij} E_Q \left(\exp\left(\int_t^T \mu_j(X_u) - R(X_u)\,\mathrm{d}u \right) \middle| \mathscr{F}_t \right), \tag{4.1.3}$$

where $\beta_{ij} = -b_{ij}b_{jK}^{-1}$, and b_{ij} is the (i,j)-th element of the matrix B, as well as b_{jK}^{-1} is the (j,K)-th entry of the matrix B^{-1}.

Proof. Recall that conditionally on X, the probability of default before T for a company in initial rating class i is equal to the (i,K)-th element of $P_X(t,T)$, which is, according to Lemma 4.2, of the form

$$P_X(t,T)_{i,K} = \sum_{j=1}^{K} b_{ij} \exp\left(\int_t^T \mu_j(X_u)\,\mathrm{d}u \right) b_{jK}^{-1}.$$

[17] Here, it is implicitly assumed that the eigenvectors of the generator matrix process are constant, which is a strong assumption.

Since the rows of A_X sum up to 0 and the last row of A_X is 0, one may check that $b_{iK}b_{KK}^{-1} = 1$. Hence, the probability of survival can be written as

$$1 - P_X(t,T)_{i,K} = \sum_{j=1}^{K-1} \beta_{ij} \exp\left(\int_t^T \mu_j(X_u)\,\mathrm{d}u\right).$$

Combining this with the general pricing formula in Theorem 4.1, we obtain our claim. (Lando, 1998, p. 112) □

Thus, in Theorem 4.3, we have expressed the zero-coupon bond price as a linear combination of conditional expected values. Hence, using the machinery of affine term structure models (that is, letting μ and R be affine functions of diffusions with affine drift and volatility), we can hope to obtain a closed-form expression for the bond price.
It is important to emphasize that the assumption of having a zero-coupon bond with zero recovery rate in Theorem 4.3 was nothing more than a simplification. Indeed, assuming a fractional recovery of market value (that is, applying the RM assumption discussed earlier), we may easily add nonzero recovery rates to the model by thinning the transition intensity processes (that is, replacing μ with $\mu(1-\phi)$ where ϕ is the fractional recovery rate of market value). In addition, coupon bonds in this framework may be priced in a straightforward manner: by summing up the prices of coupons and the principal payment.
What is still left from Lando (1998)'s model is to show how it is possible to explicitly define the state variables and calibrate the model by fitting it to credit spreads and credit spread sensitivities. Lando (1998) provides an illustrative implementation: he assumes that only one state variable governs the transition intensities and that is the process of the default-free short rate, for which he assumed a Vasicek model of the form

$$\mathrm{d}r_t = \alpha(\theta - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t,$$

where, as is well-known, W_t is a standard Brownian motion. Furthermore, he assumed that the connection between the transition intensities and the state variable is of the form of a linear combination, that is

$$\mu_j(r_t) = \gamma_j + \kappa_j r_t, \qquad j = 1, \dots K-1.$$

He then took an estimated empirical generator matrix as given and assumed that the matrix constructed from the eigenvectors of this empirical generator matrix is also the matrix created from the eigenvectors of the generator matrix process $A_X(t)$. Then, he determined the parameters γ_j and $\kappa_j, \quad j = 1, \dots K-1$ in a way that observed spot credit spreads and their sensitivities to changes in r could be matched. He chose this calibration procedure since it produces convenient linear systems of equations for obtaining γ and κ. This is illustrated in the following proposition. (Lando, 1998)

Proposition 4.4. *Define the spot spread for rating class i as*

$$s^i(r_t) = \lim_{T\to t}\left(-\frac{\partial}{\partial T}\log v^i(t,T)\right) - r_t. \tag{4.1.4}$$

Then:

$$s^i(r_t) = -\sum_{j=1}^{K-1} \beta_{ij}\mu_j(r_t) \tag{4.1.5}$$

$$\frac{\partial}{\partial r_t} s^i(r_t) = -\sum_{j=1}^{K-1} \beta_{ij}\kappa_j. \tag{4.1.6}$$

Proof.

$$-\frac{\partial}{\partial T}\log v^i(t,T) = \frac{-1}{v^i(t,T)}\frac{\partial}{\partial T}v^i(t,T) =$$
$$= \frac{-1}{v^i(t,T)}\sum_{j=1}^{K-1}\beta_{ij}E_Q\left((\gamma_j + (\kappa_j - 1)r_T)\exp\left(\int_t^T \gamma_j + \kappa_j r_u - r_u\,\mathrm{d}u\right)\middle|\mathscr{F}_t\right).$$

This, using the dominated convergence theorem, converges in case $T \to t$ to

$$\sum_{j=1}^{K-1} -\beta_{ij}(\gamma_j + (\kappa_j - 1)r_t),$$

from which, applying that $\sum_{j=1}^{K-1}\beta_{ij} = 1$, we obtain

$$s^i(r_t) = -\sum_{j=1}^{K-1}\beta_{ij}(\gamma_j + \kappa_j r_t).$$

Finally, differentiating the above formula, we get (Lando, 1998, pp. 115-116)

$$\frac{\partial}{\partial r_t} s^i(r_t) = -\sum_{j=1}^{K-1} \beta_{ij}\kappa_j.$$

□

Thus, if we observe spot credit spreads at the initial time for every rating class and we estimate their sensitivity to the short rates, we are able to calibrate our model.

Lando (1998)'s model is essentially a pricing model, which yields some points of extension if we want it to work as a model for risk management. First of all, we have not yet looked at the model as a portfolio credit risk model. This means that the output of our model is not simply a bond price, but rather a distribution of the future portfolio value. Thus, for every simulated path of the state variable processes, we will need both the simulated credit ratings for each of our exposures in one year and the prices of our bonds in one year given the state variable scenario. This way we will be able to determine the portfolio value given each simulated path, hence, its distribution.

We have not yet said anything about the dependence structure of our model. However, it is easily visible that if we extend Lando (1998)'s model with more state variables and also allow them to correlate, we will obtain a dependence structure of conditional independence given the realization of the state variables. More specifically, taking the example which was given at the discussion about the conditional independence structure of intensity models, we may rephrase equation 3.2.1 to:

$$\mu_i(X_t) = \gamma_i + \sum_{j=1}^{p} \kappa_{ij} X_{j,t}^{syst} + \nu_i X_{i,t}^{id}, \qquad i = 1, \dots m.$$

This is the most general version of the model of this kind,[18] and it could only be computable if we had individual credit spread series for each obligor at our disposal. Assuming that we only possess credit spread series for the rating classes, we can build a model in which various state variables might be included, but they can only be calibrated to the credit spread series of the rating classes:

$$\mu_j(X_t) = \gamma_j + \sum_{k=1}^{p} \kappa_{jk} X_{k,t}^{syst}, \qquad j = 1, \dots K-1.$$

Therefore, given the rating classes of individual obligors, their defaults will be independent. A version of this restricted model will be implemented in the next section. Note also that given this specification of the dependence structure, our model is capable of the allowance for market risk variables and that our resulting transition intensities will be point-in-time estimates of the migration probabilities.

Lando (1998)'s model, being a pricing model, was built under the equivalent martingale measure. However, as our model is used for risk management purposes, we need to keep track of the martingale measure as well as the physical measure, since the use of the martingale measure is necessary for pricing, but for the migration simulation, and eventually for determining the distribution of the one-year portfolio value, the physical measure is needed. For identifying the appropriate way of changing measures, let us recall that although Lando (1998) has built his model under the martingale measure, he started from the *empirical generator matrix*, that is the generator matrix under the physical measure, when illustrating his model via calculations. This means he performed a measure change by starting off from state variables calibrated under the martingale measure, and determined the model parameters essentially from the connection between the series of state variables and the credit spread series. However, when identifying the γ and κ parameters we do not need the assumed (and calibrated) processes of state variables, only their observed values. This means that if we parameterize the state variables according to the physical measure we will be able to obtain a model under this measure. The exact methods of calibrating the state variable processes under the different measures will be explained in detail in the next section.

4.2 Implementing the model

In order to illustrate that the model detailed in the previous section is feasible in practice, in this section we exhibit the implementation of a simple specification of the model. Firstly, we present our additional restricting assumptions and the reasoning behind them. More specifically, we

[18] Of course, we could change the specification to allow different copulas than the independence copula, or to incorporate interacting intensities. However, we leave this opportunity for further research.

will discuss the form of the state variable processes we used, the derivation of an analytical formula for the zero-coupon bond price in this special case, and also our method of calibrating the state variable processes to the physical as well as to the martingale measure. Finally, after reviewing the data used, we will present the results.

Lando (1998) gives an illustrative implementation of his pricing model with one state variable process: the risk-free interest rate, for which he assumed a Vasicek model. Although this way interest rate risk is taken into account in the results, we did not account for the general market movements which could fundamentally influence transition and default intensities. Allowing for globally dominant state variables is not only desirable for grasping the current deviation of the intensities from the historical average, but also for determining common factors conditional on which transitions are independent according to the assumed dependence structure. An obvious choice for a common market factor (which is also possible to be observed daily) would be a market index such as S&P 500, for which it is typical to assume that it follows a geometric Brownian motion. Similarly, the literature (Janosi, Jarrow and Yildirim (2002), Jarrow (2001), and Jarrow and Turnbull (2000)) tries to incorporate the evolution of a market index through its Brownian motion part (possibly correlated with the risk-free interest rate), which is interpreted as a measure of the cumulative excess return per unit of risk on the market index above the spot interest rate (Janosi et al., 2002; Jarrow, 2001). That is, they specify the stochastic structure of the transition intensities as

$$\mu_j(X(t)) = \gamma_j + \kappa_j r(t) + \nu_j Z(t),$$

where the dynamics of the state variable processes are defined by

$$\begin{aligned} \mathrm{d}r(t) =& [\theta - \alpha r(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}W_1^Q(t) \\ \mathrm{d}Z(t) =& \rho\,\mathrm{d}W_1^Q(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2^Q(t), \end{aligned}$$

and where the Wiener processes are under the martingale measure denoted by Q. (The correlation between the two processes is represented with the help of the well-known Cholesky decomposition.) However, with this specification the price of a zero-coupon bond can be easily bigger than one, moreover, the more distant the maturity of the bond, the higher its price, which are trivially faulty results. These observations probably stem from the fact that the variance of the Wiener process increases with time, which is also reflected in the variance of $\int_t^T \mu_j(X(u)) - r(u)\,\mathrm{d}u$. However, the zero-coupon bond's price, according to Theorem 4.3, can be written as a linear combination of expressions of the form

$$E_Q\left(\exp\left(\int_t^T \mu_j(X(u)) - r(u)\,\mathrm{d}u\right)\right).$$

It is easy to see that the exponent is a normally distributed random variable. However, it is well-known that for a normal random variable Y with mean m_Y and variance σ_Y, $E\left(e^Y\right) = e^{m_Y + 1/2\sigma_Y}$. With σ_Y increasing gradually, this will definitely be bigger than one when increasing the maturity. Thus, in order to arrive at plausible zero-coupon bond prices, we rather adhere to the use of mean-reverting processes, with asymptotically finite variance. In particular, we apply a model with two correlated Ornstein-Uhlenbeck processes, which is the simplest specification when one wants to take the market developments into account and also would like to work with stochastic interest rates. Since, apart from interest rates, currency exchange rates are also popularly modelled by Ornstein-Uhlenbeck processes, our second state variable will be an exchange rate. More specifically, we let the intensities be driven by the following specification:

$$\mu_j(X(t)) = \gamma_j + \kappa_j r(t) + \nu_j q(t), \tag{4.2.1}$$

where

$$\mathrm{d}r(t) = [\theta_1 - \alpha_1 r(t)]\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_1^Q(t) \tag{4.2.2}$$

$$\mathrm{d}q(t) = [\theta_2 - \alpha_2 q(t)]\,\mathrm{d}t + \sigma_2\rho\,\mathrm{d}W_1^Q(t) + \sigma_2\sqrt{1-\rho^2}\,\mathrm{d}W_2^Q(t). \tag{4.2.3}$$

Similarly as before, with the help of the Cholesky decomposition, we assume ρ correlation between the processes. Based on Brigo and Mercurio (2006), in this case we derive a closed-form expression for the zero-coupon bond price in the Appendix.
How is it possible to calibrate the parameters of these mean-reverting processes? The answer to this question depends on the measure according to which we specify our processes. Lando (1998)'s model could have been built wholly under the equivalent martingale measure, since there pricing was the only objective. However, in order to obtain the distribution of the future portfolio value, we are required to keep track of both the physical and the martingale measure. As argued in the end of the previous section, our whole model can be specified under either measure by calibrating the state variable processes under that measure. A method for calibrating a Vasicek model to historical data is exhibited in Brigo, Dalessandro, Neugebauer and Triki (2008). They express the discrete time version of the solution of the Ornstein-Uhlenbeck SDE as an AR(1) process, and provide the maximum likelihood estimation of its parameters.
For calibrating the Vasicek model under the martingale measure, that is calibrating it to market prices, we should either determine the so-called market price of risk process and transform the calibrated parameters under the physical measure to the parameters under the martingale measure, or calibrate the risk-neutral parameters directly from market prices. Given the market price of risk, and the well-known Vasicek model under the real-world measure with the following notation:

$$\mathrm{d}r(t) = \alpha[\theta - r(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \tag{4.2.4}$$

the parameters under the martingale measure can be derived according to the formulas presented in Vasicek (1977, cited by Gatzert and Martin (2012)):

$$\hat{\alpha} = \alpha, \qquad \hat{\theta} = \theta - \frac{\lambda\sigma}{\alpha} \qquad \text{and} \qquad \hat{\sigma} = \sigma,$$

where risk-neutral parameters are denoted by hats, and λ is the market price of risk process. However, the market price of risk process is unobservable and its determination is rather problematic. Ahmad and Wilmott (2007) and Hull, Sokol and White (2014) offer two procedures for estimating it.

Ahmad and Wilmott (2007) estimate a value for the market price of risk process for each day by using the slope of the yield curve at the short end at that day. Their estimated market price of risk time series varies wildly, mainly in the negative range, with an average of -1.2.[19] Hull et al. (2014) consider this estimate to be too large, which leads to implausible conclusions about the real world probability distribution of interest rates at longer horizons. Therefore, they apply a different technique for the estimation of λ, by comparing the long-run average short-term interest rate r_0 and the long-run average instantaneous futures rate $F(T)$ for maturity T. According to the arguments of Hull et al. (2014), a good approximation for the market price of risk for a certain maturity could be:

$$\lambda_T = -\frac{F(T) - r_0}{\sigma T}.$$

With this method, they obtained similar results to that of Ahmad and Wilmott (2007) for short maturities. However, their longer-term estimates are considerably smaller in absolute value, at about -0.26.

The other approach for calibrating the Vasicek model under the martingale measure is to determine the risk-neutral parameters directly from market prices. Cuchiero (2006) suggests a way of calibrating the Vasicek model to the current term structure. She expresses the term structure implied by the Vasicek model as

$$R(0, T_n) = \frac{-A(0, T_n)}{T_n} + \frac{B(0, T_n)}{T_n} r(0),$$

where

$$B(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}, \qquad \text{and}$$
$$A(t, T) = [B(t, T) - (T - t)] \left(\theta - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha} B(t, T)^2,$$

and the Vasicek model is parameterized as in 4.2.4, in addition, $r(0)$ is the point of the yield curve belonging to the shortest maturity (Cuchiero, 2006). The vector of yields implied by the model is then fitted to the vector of yields obtained from the market, by minimizing the sum of squared deviations with changing α, θ and σ. We used this approach for determining the Vasicek model parameters under the martingale measure in case of the short rate process. For the exchange rate process, however, no observable “yield curve” is at our disposal. Therefore, using the estimated parameters for the short rate process under both measures, we calculated the implied market price of risk (which resulted to be equal to -0.29, consistently with our expectations based on Hull et al. (2014)), and used this implied value for determining the exchange rate process parameters under the martingale measure. Trivially, this procedure

[19] In case of the market price of interest rate risk, λ is correctly expected to be negative, because if we assume investors to be risk-averse, the drift of bond prices in the real world must be higher than in the risk-neutral world. Therefore, the drift of interest rates including the short rate will be lower in the real world than in its risk-neutral counterpart.

implicitly assumes that the market price of risk for interest rates is the same as for exchange rates. This is a strong assumption, and it can be hopefully relaxed in a future work.

After explaining the additional specifications and restricting assumptions applied in the implemented version of the model, let us turn to the introduction of the data used. First and foremost we needed data regarding the fixed income portfolio of a representative life insurance company. Solvency Analytics has kindly provided to us a detailed compilation of a sample balance sheet of a Swiss life insurer, which we used for the determination of the future portfolio distribution in this study. Therefore, we have tried to collect market data relevant for a Swiss life insurance company. Accordingly, for the calibration of the parameters of the interest rate process, we used the Swiss franc LIBOR time series.[20] In the same vein, the exchange rate process has been calibrated to the USD/CHF time series.

Following the calibration methodology of Lando (1998), which was presented in Proposition 4.4, we needed credit spread time series for each rating category for the determination of the coefficients in the linear combinations 4.2.1. The only such time series we found were the Bank of America Merrill Lynch US Corporate Option-Adjusted Spread (OAS) series, from the FRED Economic Database. These spreads are calculated as the difference between a computed OAS index of all bonds in a given rating category and a spot Treasury curve.[21]

Regarding the empirical generator matrix, we have used the one presented in Jarrow et al. (1997), which has also been applied in Lando (1998). Although this is not an up-to-date estimated transition matrix, we only need a benchmark empirical generator matrix which contains through-the-cycle migration intensities published by rating agencies, and which can be modified according to the simulated intensity processes (as it was seen in Lemma 4.2), thus in our opinion this matrix is perfectly appropriate for our purposes.[22]

Our sample fixed income portfolio also includes sovereign bonds. Hence, a separate transition matrix would be desirable to be used for sovereigns, since there is no reason that sovereign rating transitions should occur according to the same process as corporate rating transitions. Therefore, Gatzert and Martin (2012) use an estimated sovereign one-year transition matrix, too. However, in our model, a sovereign transition matrix (or, more precisely, its empirical generator matrix) could only be correctly applied if there were sovereign credit spread time series for each rating grade at our service as well.[23] However, we could not obtain this kind

[20]Since the sample balance sheet is dated to 20 November 2014, our daily time series data end at this date (and start in 1998). The series are obtained from the FRED Economic Database, operated by the Federal Reserve Bank of St. Louis.

[21]It was a necessity to use US corporate data due to the lack of one better for our purposes (e.g. European corporate data). Nevertheless, it can be hoped that it does not generate a huge bias in the results since it is only used for calibration of the sensitivities applied in the computations based on Proposition 4.4.

[22]Alternatively, we could also have used e.g. the one-year transition matrix presented in Gatzert and Martin (2012), after transforming it to a generator matrix. The method for transformation is exhibited in Jarrow et al. (1997).

[23]We have tried to apply the corporate credit spread series for calibrating the time-inhomogeneous Markov process of rating transitions based on the sovereign one-year transition matrix of Gatzert and Martin (2012), too, but this has given implausible results.

of data, hence we refrained from the use of a separate transition matrix for sovereigns. Since in the benchmark sovereign transition matrix, there was much less probable for higher rating grades to jump to other categories than the ones adjacent to them, with the above detailed imprecise specification we have probably somewhat overestimated default risk of sovereigns.

The only specifying assumption we have not yet touched is about the recovery rates. As previously discussed, the model uses the recovery of market value assumption. In a more general framework, we could have let the recovery rates to be stochastic, following e.g. a beta distribution with different means for different seniority classes, as is recommended in Gupton et al. (1997). However, since we do not have information about the seniority classes each bond in our sample portfolio is assigned to, we decided to keep this part of the model as simple as possible, and used a 50% recovery rate for all bonds.

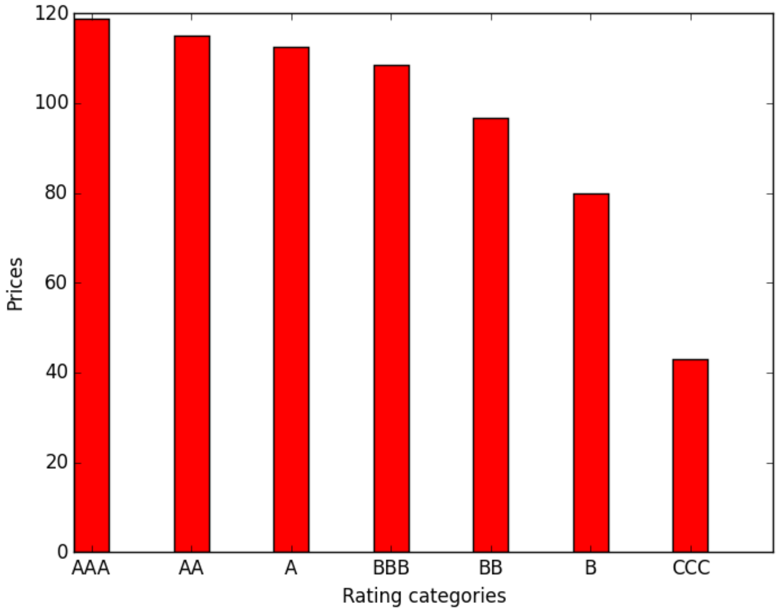
Keeping this in mind, let us have a look at the results! The calculations were performed in Python. Given that the applied stochastic structure is a key feature of the model, it is particularly interesting whether the bond prices calculated with the model are in line with our a priori intuitions. In order to investigate this question, let us present two figures. Figure 9 exhibits prices of bonds which are identical in every feature except the rating category, that is the credit risk of their issuer.[24] The results presented in this figure coincide with our expectations: the riskier the issuer, the less expensive the bond, and this relation is amplified when stepping to ratings closer to default. In addition, this figure underpins the fact that, since significant differences may be present between the prices of bonds due to the rating grades of their issuers, modelling not only defaults but also rating migrations is highly important.

Another confirmation of our results is shown in Figure 10. Here, a line displays prices of bonds paying a 1.5% coupon annually, with ascending maturities ranging from 2018 to 2036 in steps of two years. Different lines mean bonds with differently rated issuers, as the legend shows. These results are in our opinion also consistent with our expectations: the high-grade bonds are sold at a premium, which is generally the higher the longer the maturity, with a slow increase. In contrast, riskier bonds are sold at a discount. This is because investors require high yields for bearing higher default risk, hence higher expected returns (discount rates) lead to lower bond prices. This effect is trivially the stronger the longer the bond's maturity. On the other hand, in case repayment is considered so safe that a yield lower than the coupon payment is required, the bond will be traded at a premium. In our figure, the so-called investment grade bonds are sold at a premium, and the so-called junk bonds at a discount. (Obviously, this is not a general phenomenon, but is highly dependent on the size and frequency of the coupon payments, as well as on the expected yields.)

After we made sure that our assumed stochastic structure gives intuitive results when implied in practice, we are allowed to calculate an estimate for the future distribution of our fixed income portfolio. In order to obtain the estimated distribution, we simulated 2000 realizations of the state variable processes, starting at the beginning of our forecast horizon, that is on 20

[24]The bonds mature on 20 November 2031, and are paying a 2% coupon at an annual frequency.

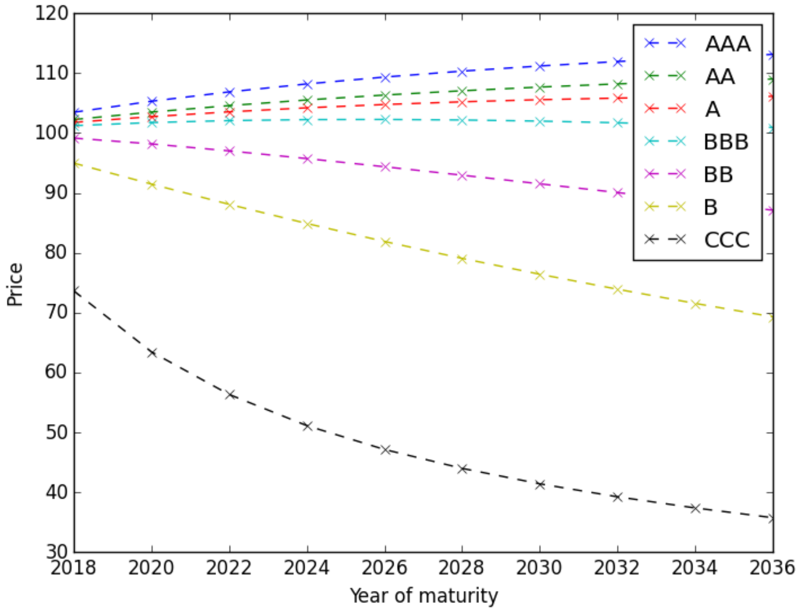
Figure 9: *Prices of a bond with different credit ratings*



Source: Own calculation.

November 2014, under both the physical and the martingale measures. Given a realization, it is obvious to calculate both the intensity processes and the series of generator matrices, according to Lemma 4.2. Then, since our aim is to determine the one-year VaR of the fixed income portfolio required by Solvency II, we simulated rating transitions within the first year, using the initial credit rating of each issuer in the portfolio, and the series of generator matrices under the physical measure. After the simulated rating migrations and the ratings in one year have been known, we priced each bond (that is, each bond's remaining cashflow elements after one year) according to the new rating category of its issuer, using the series of generator matrices under the martingale measure. After this step, it is straightforward to arrive at the simulated portfolio value in one year. Repeating these steps 2000 times we could obtain an estimation for the future portfolio distribution, which is depicted in Figure 11.
The estimated distribution exhibits long downside tails, as expected, which is natural in a fixed income portfolio where frequent small gains (planned payment of coupons and face value) are mixed with occasional large losses (default). From the estimated future portfolio distribution we calculated a one-year VaR at 99.5% confidence level, according to the instructions of Solvency II, which resulted to be equal to a loss of about 10% of the initial portfolio value (there was a little oscillation around this value in different simulations). Of course, this value should not be considered as a final number, because we implemented only a simplified version of the model framework, which can easily yield more complicated, but also more realistic specifications. This is particularly true regarding the dependence structure, as here only an implicit conditional independence has been used. When assuming more realistic dependence structures, the

Figure 10: *Prices of bonds with ascending maturities*



Source: Own calculation.

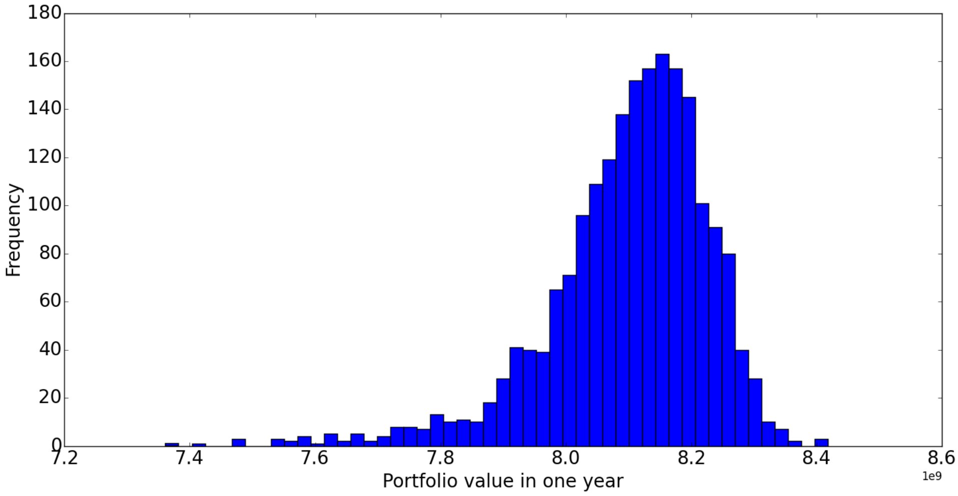
probability of joint defaults will likely increase, which can lower the VaR measure. Hence, our estimation can only be considered as a first, highly simplistic estimation of the portfolio VaR. However, the presented framework provides an opportunity for more precise specifications.

5 Conclusion

The introduction of Solvency II, an economic risk-based regulatory framework, will fundamentally change the requirements for the risk and solvency assessment of insurance companies in the European Economic Area. In order to comply with the new regulatory requirements covering all relevant types of risks of an insurance undertaking, insurers are not only allowed to implement the method of the standard approach stipulated in legislation, and calibrated rather conservatively to the features and risks of an average European insurance company, but are also permitted to develop their own internal models even only for some risk categories, with the use of which – after its official approval by supervisory bodies – they may be able to significantly reduce their Solvency Capital Requirement. Hence, insurers are highly motivated to build models which precisely address their specific risks. In this paper, we presented a possible internal model framework for credit and market risk, which could be applied to the fixed income portfolio of an insurance undertaking.

The main objective of the model was to calculate a one-year VaR measure at the 99.5% confidence level for the fixed income portfolio, according to the instructions of the Solvency II regulation, with possibly addressing as many risk factors as possible, reflecting the holistic ap-

Figure 11: *Distribution of the portfolio value in one year*



Source: Own calculation.

proach of the regulation. In particular, this thesis had the specific goal of developing a model in which credit and market risk can be incorporated simultaneously, in contrast with their often separate treatment both in regulatory frameworks and in practical models. In addition, providing an opportunity for including complex dependence structures among exposures in the model was also of high importance.

In order to determine the appropriate modelling framework, we reviewed the most important types of credit risk models discussed in the literature and evaluated them in terms of their applicability for this problem. Finally, we have chosen the framework presented in Lando (1998). This is a reduced-form credit risk model, in which we do not try to explain the occurrence of defaults explicitly, but treat the time of default as a random time which can be described with the help of a so-called intensity process. The enormous advantage of this modelling approach is that prices of zero-coupon bonds can be formulated in a way that for their computation we can apply the entire machinery of default-free term structure modelling, this way tracing back the pricing of defaultable bonds to already well-known methodologies. The model offers a simple way of incorporating recovery rates, furthermore, not only defaults but also migrations among credit ratings can be modelled. We have allowed for this feature since credit ratings have been an important input for the standard approach, too, recognizing the fact that taking ratings grades into account can significantly improve the risk assessment of a particular bond. The speciality of the chosen approach is that it allows the stochastic specification of the rating transition process. More specifically, the generator matrix whose elements determine the transition process, are made dependent upon various stochastic state variables, thus – in case of choosing the appropriate state variables in a watchful manner –, it not only links credit risk with market risk, but also maps economic cycles, current market trends to the matrix process representing credit risk, moreover, provides also plenty of opportunities for constructing realistic dependence structures among exposures. To apply his framework for risk management purposes, we extended Lando (1998)'s model, which was originally built for pricing, according to the considerations detailed in the thesis.

In order to present the applicability of the framework in practice, we performed the model calculations for the asset side fixed income portfolio of a sample Swiss life insurance company, using a simple specification. In the stochastic structure of the implemented model, we assumed two state variable processes modelled by Vasicek models, with correlation between them. One of them has been calibrated to a risk-free interest rate, the other to a relevant currency exchange rate. This simply specified model has given results consistent with our a priori expectations, which raises hope for obtaining real partial internal models for credit and market risk by choosing more realistic specifications in this framework. However, these more complex assumptions, as well as building further risk factors (such as e.g. equity, property or perhaps liquidity risk) into the model, and comparing the performance of the model with the standard approach for different portfolios, are left for future research.

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Appendix - Derivation of the zero-coupon bond price

Let us recall from the main text that we are looking for

$$E_Q\left(\exp\left(\int_t^T \gamma_j + (\kappa_j - 1)r(u) + \nu_j q(u)\,\mathrm{d}u\right)\right),$$

where

$$\begin{aligned}\mathrm{d}r(t) =& [\theta_1 - \alpha_1 r(t)]\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_1^Q(t)\\ \mathrm{d}q(t) =& [\theta_2 - \alpha_2 q(t)]\,\mathrm{d}t + \sigma_2\rho\,\mathrm{d}W_1^Q(t) + \sigma_2\sqrt{1-\rho^2}\,\mathrm{d}W_2^Q(t).\end{aligned}$$

For simplicity, we will set aside the j subscripts, and also incorporate the discount factor into κ. In addition, the measure according to which we determine the expected value, is now irrelevant (more precisely, it is incorporated in the parameters of the processes), thus we also drop its subscript. That is, we will try to express

$$E\left(\exp\left(\int_t^T \gamma + \kappa r(u) + \nu q(u)\,\mathrm{d}u\right)\right).$$

Firstly, let us try to determine the distribution of the exponent. The solution of the SDE for $r(t)$ is well-known from the standard Vasicek model:

$$r(t) = r(s)e^{-\alpha_1(t-s)} + \int_s^t e^{-\alpha_1(t-u)}\theta_1\,\mathrm{d}u + \int_s^t e^{-\alpha_1(t-u)}\sigma_1\,\mathrm{d}W_1(u).$$

Hence, with the help of the Fubini theorem,

$$\begin{aligned}\int_t^T r(u)\,\mathrm{d}u =& r(t)\int_t^T e^{-\alpha_1(u-t)}\,\mathrm{d}u + \int_t^T\int_t^u e^{-\alpha_1(u-v)}\theta_1\,\mathrm{d}v\,\mathrm{d}u +\\ &+ \int_t^T\int_t^u e^{-\alpha_1(u-v)}\sigma_1\,\mathrm{d}W_1(v)\,\mathrm{d}u =\\ =& \frac{r(t)}{\alpha_1}\left(1-e^{-\alpha_1(T-t)}\right) + \frac{\theta_1}{\alpha_1}\int_t^T\left(1-e^{-\alpha_1(T-v)}\right)\mathrm{d}v +\\ &+ \frac{\sigma_1}{\alpha_1}\int_t^T\left(1-e^{-\alpha_1(T-v)}\right)\mathrm{d}W_1(v) =\\ =& \frac{r(t)}{\alpha_1}\left(1-e^{-\alpha_1(T-t)}\right) + \frac{\theta_1}{\alpha_1}(T-t) - \frac{\theta_1}{\alpha_1}\frac{1}{\alpha_1}\left(1-e^{-\alpha_1(T-t)}\right) +\\ &+ \frac{\sigma_1}{\alpha_1}\int_t^T\left(1-e^{-\alpha_1(T-v)}\right)\mathrm{d}W_1(v).\end{aligned}$$

In the same vein,

$$\int_t^T q(u)\,\mathrm{d}u = \frac{q(t)}{\alpha_2}\left(1-e^{-\alpha_2(T-t)}\right)+\frac{\theta_2}{\alpha_2}(T-t)-\frac{\theta_2}{\alpha_2}\frac{1}{\alpha_2}\left(1-e^{-\alpha_2(T-t)}\right)+$$
$$+\frac{\sigma_2}{\alpha_2}\rho\int_t^T\left(1-e^{-\alpha_2(T-v)}\right)\mathrm{d}W_1(v)+\frac{\sigma_2}{\alpha_2}\sqrt{1-\rho^2}\int_t^T\left(1-e^{-\alpha_2(T-v)}\right)\mathrm{d}W_2(v).$$

Now it is already visible that $\int_t^T \kappa r(t)+\nu q(t)\,\mathrm{d}u$ is normally distributed. Its mean:

$$E\left(\int_t^T \kappa r(u)+\nu q(ut)\,\mathrm{d}u\right)=\kappa\left(r(t)-\frac{\theta_1}{\alpha_1}\right)\frac{1-e^{-\alpha_1(T-t)}}{\alpha_1}+\kappa\frac{\theta_1}{\alpha_1}(T-t)+$$
$$+\nu\left(q(t)-\frac{\theta_2}{\alpha_2}\right)\frac{1-e^{-\alpha_2(T-t)}}{\alpha_2}+\nu\frac{\theta_2}{\alpha_2}(T-t).$$

For its variance, we need to use the Ito isometry and the independence of the Wiener processes:

$$V(t,T)=D^2\left(\int_t^T \kappa r(u)+\nu q(u)\,\mathrm{d}u\right)=$$
$$=D^2\Bigg[\kappa\frac{\sigma_1}{\alpha_1}\int_t^T\left(1-e^{-\alpha_1(T-v)}\right)\mathrm{d}W_1(v)+\nu\frac{\sigma_2}{\alpha_2}\rho\int_t^T\left(1-e^{-\alpha_2(T-v)}\right)\mathrm{d}W_1(v)+$$
$$+\nu\frac{\sigma_2}{\alpha_2}\sqrt{1-\rho^2}\int_t^T\left(1-e^{-\alpha_2(T-v)}\right)\mathrm{d}W_2(v)\Bigg]=$$
$$=D^2\Bigg[\int_t^T\kappa\frac{\sigma_1}{\alpha_1}\left(1-e^{-\alpha_1(T-v)}\right)+\nu\frac{\sigma_2}{\alpha_2}\rho\left(1-e^{-\alpha_2(T-v)}\right)\mathrm{d}W_1(v)+$$
$$+\int_t^T\nu\frac{\sigma_2}{\alpha_2}\sqrt{1-\rho^2}\left(1-e^{-\alpha_2(T-v)}\right)\mathrm{d}W_2(v)\Bigg]=$$
$$=\int_t^T\left[\kappa\frac{\sigma_1}{\alpha_1}\left(1-e^{-\alpha_1(T-v)}\right)+\nu\frac{\sigma_2}{\alpha_2}\rho\left(1-e^{-\alpha_2(T-v)}\right)\right]^2\mathrm{d}v+$$
$$+\int_t^T\left[\nu\frac{\sigma_2}{\alpha_2}\sqrt{1-\rho^2}\left(1-e^{-\alpha_2(T-v)}\right)\right]^2\mathrm{d}v.$$

From this, simple integration leads to

$$V(t,T)=\kappa^2\frac{\sigma_1^2}{\alpha_1^2}\left[(T-t)-2\left(\frac{1-e^{-\alpha_1(T-t)}}{\alpha_1}\right)+\frac{1-e^{-2\alpha_1(T-t)}}{2\alpha_1}\right]+$$
$$+\nu^2\frac{\sigma_2^2}{\alpha_2^2}\left[(T-t)-2\left(\frac{1-e^{-\alpha_2(T-t)}}{\alpha_2}\right)+\frac{1-e^{-2\alpha_2(T-t)}}{2\alpha_2}\right]+$$
$$+2\kappa\nu\rho\frac{\sigma_1}{\alpha_1}\frac{\sigma_2}{\alpha_2}\left[(T-t)-\frac{1-e^{-\alpha_1(T-t)}}{\alpha_1}-\frac{1-e^{-\alpha_2(T-t)}}{\alpha_2}-\frac{1-e^{-(\alpha_1+\alpha_2)(T-t)}}{\alpha_1+\alpha_2}\right].$$

Finally, using the fact that for a normally distributed random variable Y with mean m_Y and variance σ_Y, $E\left(e^Y\right)=e^{m_Y+1/2\sigma_Y}$, and writing back the deterministic γ, we obtain the closed-form expression for our expected value:

$$E\left(\exp\left(\int_t^T \gamma+\kappa r(u)+\nu q(u)\,\mathrm{d}u\right)\right)=$$
$$=\exp\Bigg\{\gamma(T-t)+\kappa\left(r(t)-\frac{\theta_1}{\alpha_1}\right)\frac{1-e^{-\alpha_1(T-t)}}{\alpha_1}+\kappa\frac{\theta_1}{\alpha_1}(T-t)+$$
$$+\nu\left(q(t)-\frac{\theta_2}{\alpha_2}\right)\frac{1-e^{-\alpha_2(T-t)}}{\alpha_2}+\nu\frac{\theta_2}{\alpha_2}(T-t)+\frac{1}{2}V(t,T)\Bigg\}.$$